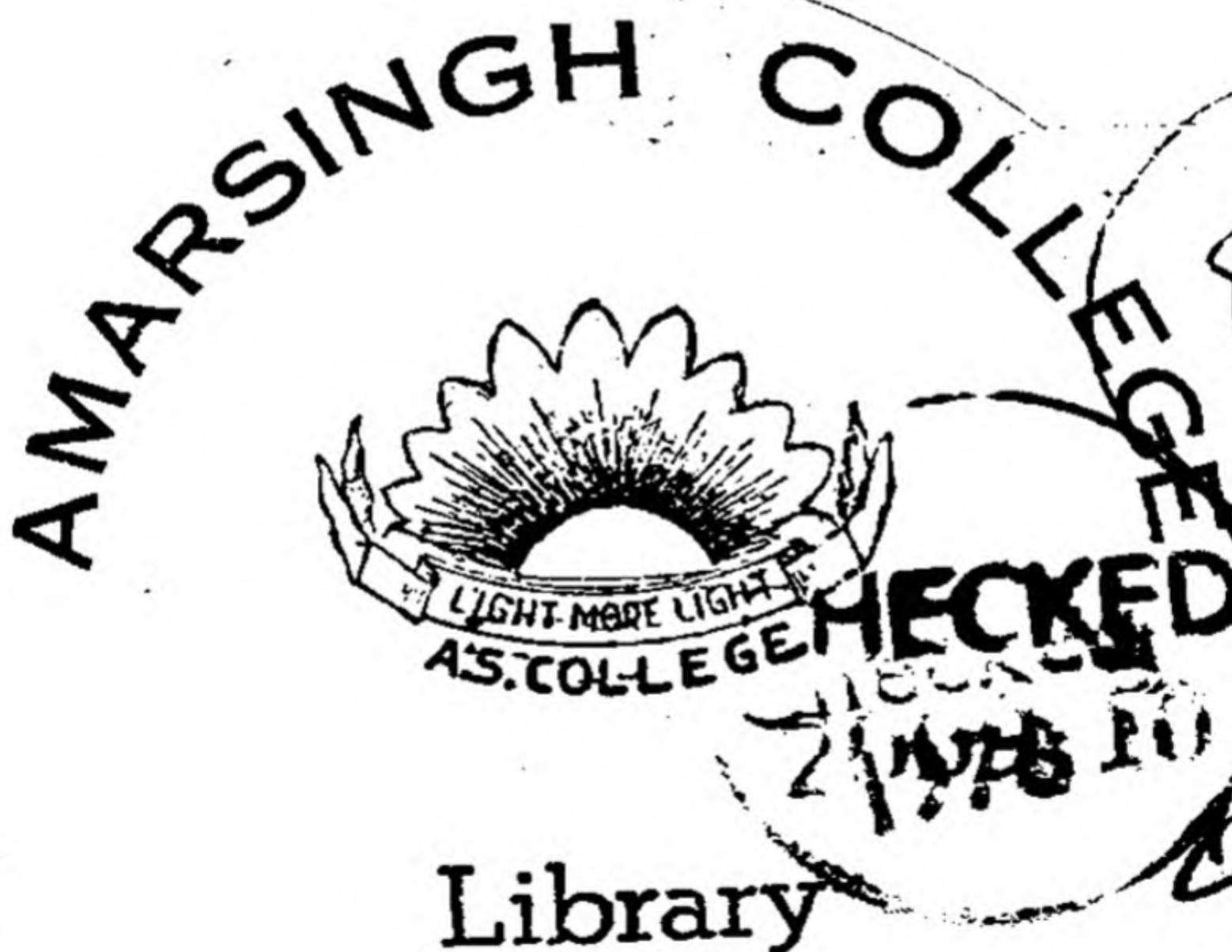


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INTERMEDIATE ALGEBRA

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CHAPTER I

QUADRATIC EQUATIONS

1.1. Degree of an Equation. An equation which when reduced to rational and integral form contains terms of n th degree in the unknown quantity, say, x and none of the higher degree, is said to be an equation of n th degree in x .

Thus $ax+b=0$, $ax^2+bx+c=0$, $ax^3+bx^2+d=0$ are equations of first, second, third degree, respectively. The equations of first, second, third, fourth and fifth degree are also known as *Linear*, *Quadratic*, *Cubic*, *Quartic* and *Quintic* respectively.

Ex. Determine the degree of the following equations :—

(i) $2x + \frac{1}{x} = 3$. (ii) $\sqrt{x} + \frac{1}{\sqrt{x}} = 9$.

(iii) $\sqrt[3]{1+x^2} = \frac{1}{x} + 3$.

Sol :

(i) $2x + \frac{1}{x} = 3$ when written with positive integral indices of x is $2x^2 + 1 = 3x$. Highest degree in x , being 2, the equation is of the second degree.

(ii) $\sqrt{x} + \frac{1}{\sqrt{x}} = 9$.

Squaring both sides, we get $x + \frac{1}{x} + 2 = 81$.

$\therefore x^2 - 79x + 1 = 0$ which is a second degree equation.

(iii) $\sqrt[3]{1+x^2} = \frac{1}{x} + 3$.

Taking cubes of both sides $(1+x^2) = \left(\frac{1}{x} + 3\right)^3$.

Multiplying by x^3 we get $x^3(1+x^2) = (1+3x)^3$.

$\therefore$ It is a fifth degree equation in x .

1.11. We are here mainly concerned with the second degree equation in x (or what is known as the Quadratic Equation), e.g.,

$$9x^2 + 12x + 11 = 0, \quad 3x + \frac{1}{x} = 9, \quad x + \sqrt{2x} = 1.$$

The equation $ax^2 + bx + c = 0$, is called the **general quadratic equation in x** and a, b, c are called **co-efficients of x^2, x** and the constant term respectively.

1.2. Solution of an Equation. The values of the unknown quantity or the variable for which the equation is satisfied are called *roots of the equation* and the process of finding them is called *solving the equation*.

Thus $x - 3 = 0$ is an equation which is satisfied when $x = 3$ $\therefore$ 3 is a root of this equation. Similarly $x^2 - 5x + 6 = 0$ is satisfied by $x = 3$ or 2. Thus 2 and 3 are roots of $x^2 - 5x + 6 = 0$.

Ex. 1. Solve $2x^2 + 11x + 15 = 0$.

Sol. Since $2x^2 + 11x + 15 = 2x^2 + 5x + 6x + 15$
 $= (2x + 5)(x + 3).$

$\therefore 2x^2 + 11x + 15 = 0$, when $(2x + 5)(x + 3) = 0$, i.e., when
 $2x + 5 = 0$ or $x + 3 = 0 \therefore$ either $x = -\frac{5}{2}$ or $x = -3$.

Hence the roots are $-\frac{5}{2}$ and -3 .

The method used above is called the **method of factors**.

Another Important Method :—

We have $2x^2 + 11x + 15 = 0$

or $x^2 + \frac{11}{2}x + \frac{15}{2} = 0$, (Divide by Co-efficient of x^2)

or $x^2 + \frac{11}{2}x + \frac{121}{16} = \frac{121}{16} - \frac{15}{2}$.

(Complete the square by adding on both sides a suitable number and shift constant to the other side.)

$$\therefore \left(x + \frac{11}{4}\right)^2 = \frac{121 - 120}{16}$$

QUADRATIC EQUATIONS

$$\left(x + \frac{11}{4}\right)^2 = \left(\frac{1}{4}\right)^2$$

$$\therefore x + \frac{11}{4} = \pm \frac{1}{4}$$

$$\therefore x = -3, -\frac{5}{2}.$$

This method is called the method of **completing the square**.
(of. Art. 1.3)

Observation. The advantage with the second method is that a quadratic can be solved even if it be not possible to find the factors of the left hand side expression. Thus e.g.,

$$3x^2 + 5x + 1 = 0.$$

Here we cannot factorise L. H. S. but dividing by 3 we get

$$x^2 + \frac{5}{3}x + \frac{1}{3} = 0.$$

$$\therefore \left(x + \frac{5}{6}\right)^2 = \left(\frac{5}{6}\right)^2 - \frac{1}{3} = \frac{13}{36}$$

$$\therefore x + \frac{5}{6} = \pm \frac{\sqrt{13}}{6}$$

$$\therefore x = \frac{-5 \pm \sqrt{13}}{6}$$

Ex. 2. Solve $\frac{a}{x-b} + \frac{b}{x-a} = 2.$

Transposing we get $\frac{a}{x-b} - 1 + \frac{b}{x-a} - 1 = 0,$

or $\frac{a+b-x}{x-b} + \frac{a+b-x}{x-a} = 0.$

$$\text{i. e. } (a+b-x) \left[\frac{1}{x-b} + \frac{1}{x-a} \right] = 0.$$

$\therefore$ either $a+b-x=0$, i.e., $x=a+b$;

or $\frac{1}{x-b} + \frac{1}{x-a} = 0$ i.e., $2x-(a+b)=0$, or $x = \frac{a+b}{2}.$

Caution : It is important to observe that if there is a factor (containing the variable) common in an equation throughout, it

should not only be cancelled but should be equated to zero also, and the values of the variable so found are also roots of the original equation. Thus in $x^3 + 5x^2 = -6x$ one root is $x=0$ and in $x^4 + 5x^3 = -6x^2$ two roots are repeated each $=0$.

Note :- Division by zero is inadmissible and the student is advised never to divide both sides of an equation by an expression which is equal to zero. This sometimes leads to absurd results as illustrated below :-

Let $x=a$.

Then $x^2=ax$

Also $x^2 - a^2 = ax - a^2$, i.e., $(x+a)(x-a) = a(x-a)$

$\therefore x+a=a$ or $a+a=a$

$\therefore 2a=a$, i.e., $2=1$.

Ex. 3. Solve $\sqrt{x+6} - \sqrt{x-6} = 2$

Here $\sqrt{x+6} = 2 + \sqrt{x-6}$.

Squaring both sides, $x+6 = x-6 + 4 + 4\sqrt{x-6}$

$\therefore 8 = 4\sqrt{x-6}$ or $2 = \sqrt{x-6}$

Squaring $4 = x-6 \therefore x=10$. (It satisfies the equation).

EXERCISE I

Solve the following equations :

1. (i) $6x^2 - 5x + 1 = 0$.

(ii) $6x^2 + x - 2 = 0$.

(iii) $x^3 + x^2 - 2x = 0$.

2. $4x^2 - 2(p+q)x + pq = 0$.

3. $\frac{3x+1}{2x+1} = \frac{x+3}{x+2}$.

4. $\frac{x}{a} - \frac{b}{x} = \frac{a+b}{x} + \frac{b^2}{ax}$.

5. $\sqrt{x+4} + \sqrt{x-4} = 4$.

6. $\frac{6x-5}{9} + \frac{3x^2-5}{4x^2+5} = \frac{2x-\frac{2}{3}}{3}$.

7. $\frac{x-2}{x-4} + \frac{x-4}{x-6} = 2$.

8. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} + \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 98$.

9. $\sqrt{3x^2+7x+5} = 0$.

10. $\frac{3x^2}{2} + 2x = 9$.

11. Solve: $\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}$. (H. S. 1949.)

12. $\sqrt{2x+6} - \sqrt{x-1} = 2$. (H. S. 1947.)

1.3. To solve the general quadratic equation.

Let $ax^2 + bx + c = 0$ (1)

be a quadratic equation where a is not zero.

Transposing c to the other side, we have

$$ax^2 + bx = -c.$$

Dividing both sides by a , the co-efficient of x^2 , we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{.....(2)}$$

Adding $\left(\frac{b}{2a}\right)^2$ [square of half the co-efficient of x in (2)] to both sides, so that L. H. S. is a complete square we get

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

or $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$

Taking the square root, we get

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{.....(3)}$$

Hence the roots of the equation are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

It shows that a quadratic equation has two roots. It will be proved later that it cannot have more than two roots.

Note.—A slightly different method is found in ‘Bij Ganita,’ of Bhaskracharya, a well-known Hindu Mathematician of the 13th century. After transposing c to the right-hand side, he multiplies the equation by $4a$ and adds b^2 to each side, thus :—

$$ax^2 + bx = -c.$$

$$4a^2x^2 + 4abx = -4ac \text{ or } 4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$\therefore (2ax + b)^2 = b^2 - 4ac \text{ or } 2ax + b = \pm \sqrt{b^2 - 4ac}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Ex. 1. Solve $6x^2 + 7x - 3 = 0$.

Dividing by 6, we have $x^2 + \frac{7}{6}x = \frac{3}{6} = \frac{1}{2}$

Adding $\left(\frac{7}{12}\right)^2$ to both sides, we get

$$x^2 + \frac{7}{6}x + \left(\frac{7}{12}\right)^2 = \frac{1}{2} + \left(\frac{7}{12}\right)^2$$

$$\text{or } \left(x + \frac{7}{12}\right)^2 = \frac{1}{2} + \frac{49}{144}$$

$$= \frac{72 + 49}{144} = \left(\frac{11}{12}\right)^2$$

$$\therefore x + \frac{7}{12} = \pm \frac{11}{12}$$

$$\therefore \text{either } x = \frac{11}{12} - \frac{7}{12} = \frac{1}{3},$$

$$\text{or } x = -\frac{11}{12} - \frac{7}{12} = -\frac{3}{2}.$$

Ex. 2. Solve $2x^2 = x(a + 2b) + a^2 + ab$.

The given equation is $2x^2 + x(-a - 2b) - (a^2 + ab) = 0$.

$\therefore$ by the formula (3), we have

$$x = \frac{-(-a - 2b) \pm \sqrt{(-a - 2b)^2 + 4 \times 2(a^2 + ab)}}{2 \times 2}$$

$$= \frac{a + 2b \pm (3a + 2b)}{4} = a + b \text{ or } -\frac{a}{2}.$$

EXERCISE II-A

Solve :

1. $10x^2 = 14 - 31x$.

2. $6x^2 + 5x + 3 = 0$.

3. $x^2 - 2bx = 1 - b^2$.

4. $\frac{1}{x-1} - \frac{1}{x+1} = 1$.

5. $\frac{a}{x} + \frac{x}{a} = \frac{9a^2 - x^2}{ax}$.

6. $\frac{a}{b+x} - \frac{2b}{a+x} = 1$.

7. $x^3 + 7x^2 - 21x - 27 = 0$.

[Hint : — Factorise] (D.B.)

$$8. \quad x^3 - 5x^2 + 6x = 0.$$

$$9. \quad \frac{x+b}{a-b} + \frac{x-3a}{a+b} = \frac{x-a}{2b}.$$

$$10. \quad \frac{b}{x-a} + \frac{a}{x+b} = \frac{a+b}{2(a-b)}$$

$$11. \quad \frac{x+1}{a^2+a} = \frac{a^2-a}{x}.$$

$$12. \quad \frac{b}{x+a} - \frac{2a}{x+b} = 1.$$

$$13. \quad \frac{x-a}{b} + \frac{a}{x-b} = 2.$$

$$14. \quad ab(1-x^2) = (ax+b)(bx+a). \quad 15. \quad \frac{x+a}{a} - \frac{a}{x+a} = \frac{3}{2}.$$

$$16. \quad \frac{x+2}{x-3} + \frac{x+1}{x-2} = 4.$$

$$17. \quad \frac{1}{x-1} + \frac{4}{x-4} = \frac{2}{x-2} + \frac{3}{x-3}.$$

(D.B.)

$$18. \quad \frac{a}{ax-1} + \frac{b}{bx-1} = a+b. \quad 19. \quad abx^2 = (a+b)^2(x-1).$$

$$20. \quad (x-2b)(x+a) + (x+b)(x-a) = 2bx - 3ab - b^2.$$

$$21. \quad x^4 - a^4 = (x^2 - a^2)(2x^2 - 15a^2).$$

$$22. \quad \frac{1}{x+b} + \frac{1}{x+2b} + \frac{1}{x+3b} = \frac{3}{x}.$$

1.31. Harder Solved Examples :—

Ex. 1. Solve : $(a+x)^{\frac{1}{3}} + (b-x)^{\frac{1}{3}} = (a+b)^{\frac{1}{3}}.$

Taking cubes of both sides, we get

$$(a+x) + (b-x) + 3(a+x)^{\frac{1}{3}}(b-x)^{\frac{1}{3}}[(a+x)^{\frac{1}{3}} + (b-x)^{\frac{1}{3}}] = a+b$$

$$\text{But } (a+x)^{\frac{1}{3}} + (b-x)^{\frac{1}{3}} = (a+b)^{\frac{1}{3}}$$

[Important Step]

Substituting this and simplifying, we get

$$3(a+x)^{\frac{1}{3}}(b-x)^{\frac{1}{3}}[(a+b)^{\frac{1}{3}}] = 0,$$

$$\text{or } (a+x)^{\frac{1}{3}}(b-x)^{\frac{1}{3}} = 0$$

$$\text{i.e., } (a+x)(b-x) = 0$$

$$\therefore x = -a \text{ and } x = b \text{ (Both satisfy the equation)}$$

Ex. 2. Solve : $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Squaring both sides, we get

$$x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$$

$$\sqrt{(x+7)(x+2)}=2x+2$$

Squaring again we get

$$(x+7)(x+2)=4(x+1)^2$$

$$\text{or } x^2+9x+14=4x^2+8x+4$$

$$\text{or } 3x^2-x-10=0$$

$$\therefore x = \frac{1 \pm \sqrt{1+120}}{6} = 2 \text{ or } -\frac{5}{3}$$

Substituting the values of x in the given equation, we find that $x=2$ satisfies it while $x=-\frac{5}{3}$ does not.

Hence $x=2$ is the only solution.

Extraneous Roots :—

It may be observed that if at any step in the process of solution of an equation its both sides are squared, some extra solutions are introduced which do not satisfy the original equation and are known as *extraneous roots*. Such solutions are generally the roots of another equation obtained by changing the sign of some of its radicals. In the above equation, for example, $-\sqrt{x+7} - \sqrt{x+2} = \sqrt{6x+13}$ is root of $\sqrt{x+7} - \sqrt{x+2} = \sqrt{6x+13}$ and now $x=2$ is its extraneous root.

Ex. 3. Solve $\frac{x-\sqrt{2x+3}}{x+\sqrt{2x+3}} = \frac{1}{2}$.

Here by componendo and dividendo we have $\frac{x+3}{\sqrt{2x}} = \frac{3}{1}$

$$\therefore x+3=3\sqrt{2x}$$

Squaring both sides, $(x+3)^2=18x$

$$\text{or } x^2+6x+9=18x$$

$$\text{i.e. } x^2-12x+9=0$$

$$x = \frac{12 \pm \sqrt{144-36}}{2} = \frac{12 \pm 6\sqrt{3}}{2} = 6 \pm 3\sqrt{3}$$

Ex. 4. Find the possible values of

$$\frac{y}{x} \text{ when } \frac{x^2 + 15xy - 4y^2}{x^2 + y^2} = 6.$$

Here let $\frac{y}{x} = m$

$$\therefore y = mx$$

Hence given equation becomes

$$\frac{x^2 + 15mx^2 - 4m^2x^2}{x^2 + m^2x^2} = 6$$

or

$$\frac{1 + 15m - 4m^2}{1 + m^2} = 6$$

or

$$1 + 15m - 4m^2 = 6 + 6m^2$$

or

$$10m^2 - 15m + 5 = 0$$

or

$$2m^2 - 3m + 1 = 0$$

$\therefore$

$$m = \frac{3 \pm \sqrt{9 - 8}}{4}$$

$$= \frac{3 \pm 1}{4} = 1, \frac{1}{2}$$

$\therefore$

values of $\frac{y}{x}$ are 1, $\frac{1}{2}$.

EXERCISE II-B

$$1. (1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}} = a^{\frac{1}{3}}.$$

$$2. (a+x)^{\frac{2}{3}} + (a-x)^{\frac{2}{3}} = 4(a^2-x^2)^{\frac{1}{3}}.$$

$$3. (l+mx)^{\frac{2}{3}} - (l-mx)^{\frac{2}{3}} = (4lmx)^{\frac{1}{3}}.$$

$$4. \sqrt{2x+8} + \sqrt{x+5} = 7.$$

(Q. 1950)

$$5. \sqrt{2x+1} + \sqrt{x+4} = 3.$$

(D. U. 1945)

$$6. \sqrt{2x+1} + \sqrt{3x+2} = \sqrt{5x+3}.$$

$$7. \text{ Show that } \sqrt{2x-5} - \sqrt{4x-9} = \sqrt{4x+7} \text{ has no root.}$$

[Hint. The roots obtained by the usual method do not satisfy the equation].

$$8. \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} = 3.$$

9. Find the two values of x for which the expression $\frac{2x}{x+1} + \frac{3(x+5)}{(x+1)(x-3)}$ has the value 7.
10. If $\frac{3x+y}{x+2y} = \frac{x+5y}{3x+y}$, find the possible values of $\frac{x}{y}$.
11. Find the values of $\frac{y}{x}$ from
- (i) $\frac{5y^2-7x^2}{5xy-6x^2} = \frac{17}{6}$. (ii) $\frac{x^2-xy+y^2}{x^2+xy+y^2} = \frac{19}{49}$.
12. Solve the equation :
- $$\frac{16}{x^{\frac{3}{2}}} + \frac{x^{\frac{1}{2}}}{2} = \frac{6}{x^{\frac{1}{2}}}$$

(H. S. B)

1.4. Application of Quadratic Equations in solving problems :—

We now give a few problems the solution of which is based on the solution of a quadratic equation.

Ex. 1. Divide a line of length b into two parts such that the rectangle contained by that line and one part may be equal to the square on the other.

Let x be one part of the line, then $b-x$ is the other part.

$$\therefore b(b-x) = x^2 \quad \text{or} \quad x^2 + bx - b^2 = 0$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 + 4b^2}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2} b.$$

The sign of square root of 5 is $+ve$ or $-ve$ according as the line is divided internally or externally.

This corresponds to an example in Geometry, of dividing a given line in **medial section**.

Ex 2. The current of a stream runs at the rate of five miles an hour. A motor boat goes ten miles up the stream and back again to its starting point in 50 minutes. Find the speed of the motor-boat in still water.

Let the speed of the motor boat in still water be x miles per hour. The time in hours taken in the journey up the

stream is $\frac{10}{x-5}$ and in the journey down the stream it is $\frac{10}{x+5}$.

$$\therefore \frac{10}{x-5} + \frac{10}{x+5} = \frac{50}{60},$$

$$\text{or } 10(x-5+x+5) = \frac{5}{6} \times (x^2-25)$$

$$\text{or } 24x = x^2 - 25 \text{ or } x^2 - 24x - 25 = 0$$

$$\text{i. e. } (x-25)(x+1) = 0.$$

$$\therefore x = 25 \text{ m. p. h.}$$

(The negative value -1 , is rejected).

EXERCISE III

1. Divide 30 into two parts such that their product is 200.

2. Of six consecutive positive integers, the product of the last three exceeds the product of the first three by 816 ; find them.

3. Divide a line 10 centimeters long internally into two parts such that the rectangle contained by the whole line and one part may be twice the square on the other part.

4. A person covers a distance of 100 miles in 2 hours. He rides 45 miles in one car and 55 miles in another which travels 10 miles per hour faster than the first. Find the speeds of the cars.

5. The area of a rectangle is 184 sq. ft. The length of the rectangle is 3 ft. 4 in. more than the breadth. Find the length and the breadth.

MISCELLANEOUS EQUATIONS

1.5. Equations reducible to Quadratic Equations.
We give here, some of the types of equations which can be reduced to quadratic equations.

1.51. First Type : *In equations of the type*
 $ax^{2n} + bx^n + c = 0,$
if x^n is put equal to z the equation becomes $az^2 + bz + c = 0$
which is a quadratic equation in z .

Ex. $x^6 - 9x^3 + 8 = 0.$

Here let $x^3 = z.$

which gives $z = 8$ or $1,$

$$\therefore z^2 - 9z + 8 = 0$$

i. e., $x^3=8$ or 1.

$\therefore x=2$ or 1. (taking real values).

1.52. **Second Type :** Equations of the form $aX + \frac{b}{X} = c$, where a, b, c are constant and X is an expression containing the variable whose values are required.

Ex. Solve : $\sqrt{\frac{x^2+2}{x^2-2}} + 6\sqrt{\frac{x^2-2}{x^2+2}} = 5.$

Let $\sqrt{\frac{x^2+2}{x^2-2}} = z \therefore z + \frac{6}{z} = 5.$

or $z^2 - 5z + 6 = 0. \therefore z = 3$ or 2.

i. e., $\sqrt{\frac{x^2+2}{x^2-2}} = 3$ or 2 or $\frac{x^2+2}{x^2-2} = 9$ or 4

which easily gives

$$x = \pm \sqrt{\frac{5}{2}} \text{ or } \pm \sqrt{\frac{10}{3}}$$

the four values of x which satisfy the given equation.

EXERCISE IV

Solve the following equations :

1. $x^4 - 5x^2 + 6 = 0.$

2. $x^8 - 20x^4 + 64 = 0.$

3. $x^6 - 9x^3 + 8 = 0.$

4. $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 = 0.$

5. $\frac{2}{x^n} - ax^{\frac{1}{n}} - 6a^2 = 0.$

6. $2^{2x} - 3 \cdot 2^{x+2} + 32 = 0.$

7. $4^{1+x} + 4^{1-x} = 10.$

8. $3^{2x} - 10 \cdot 3^x + 9 = 0.$

9. $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = 2\frac{1}{6}.$ (D.U. 1945, 47, 51

and H. S. 46)

10. $\left(x - \frac{1}{x}\right)^2 - 5\left(x - \frac{1}{x}\right) + 4 = 0.$

11. (i) $\sqrt{x^2 - 3x} = 4x^2 - 12x - 3.$

(ii) $x^2 + 2\sqrt{x^2 + 2x + 3} = 12 - 2x.$

(H. S. 1948)

12. $\left(\frac{2x+1}{x-1}\right)^4 - 10\left(\frac{2x+1}{x-1}\right)^2 + 9 = 0.$

$$13. \sqrt{\frac{2x^2+x+2}{x^2+3x+1}} + 2 \sqrt{\frac{x^2+3x+1}{2x^2+x+2}} - 3 = 0.$$

$$14. \sqrt{\frac{2x^2+1}{x^2-1}} + 6 \sqrt{\frac{x^2-1}{2x^2-1}} = 5.$$

$$15. \sqrt{\frac{3x}{x+1}} - \sqrt{\frac{x+1}{3x}} = 2. \quad 16. \frac{a+x}{b+x} + \frac{b+x}{a+x} = 2\frac{1}{2}. (P.U.)$$

1.53. Third type: Equation of the type

$$(x+a)(x+b)(x+c)(x+d)+k=0,$$

where sum of two of the quantities a, b, c, d , is equal to the sum of the other two, can be solved as shown below.

Ex. 1. Solve $(x+1)(x+5)(x+3)(x+7)=-15$.

Here $1+7=3+5$

$\therefore$ we get $[(x+1)(x+7)][(x+5)(x+3)]=-15$.

i. e., $(x^2+8x+7)(x^2+8x+15)=-15$.

Putting $x^2+8x=z$ we get

$$(z+7)(z+15)=-15$$

$$z^2+22z+120=0$$

$$z=-10 \text{ or } -12.$$

$\therefore x^2+8x=-10$ or $x^2+8x=-12$.

These give $x=-4 \pm \sqrt{6}$ or $x=-6, -2$.

Ex. 2. Solve : $(2x-7)(x^2-9)(2x+5)=91$.

Here the given equation is

$$(2x-7)(x-3)(x+3)(2x+5)=91$$

$$\text{or } (2x-7)(x+3)(2x+5)(x-3)=91$$

$$\text{or } (2x^2-x-21)(2x^2-x-15)=91$$

$$\text{Put } 2x^2-x-21=y$$

$$\therefore y(y+21-15)=91$$

$$\text{or } y^2+6y-91=0$$

$$\therefore y=7, -13.$$

$$\therefore 2x^2-x-21=7 \text{ or } 2x^2-x-21=-13$$

$$\text{i. e., } 2x^2-x-28=0 \text{ or } 2x^2-x-8=0$$

$$\therefore x=4, \frac{-7}{2} \text{ or } x=\frac{1 \pm \sqrt{65}}{4}.$$

EXERCISE V

Solve the following equations :

1. $(x+1)(x+2)(x+3)(x+4)+1=0$.

2. $x(x-1)(x+2)(x-3)+8=0$

3. $(x-7)(x-3)(x+1)(x+5)=1680$.

4. $(x+a)(x+3a)(x+5a)(x+7a)=384a^4$.

5. $(12x-1)(6x+1)(4x-1)(2x-1)=-1$.

[**Hint.** Here combine first and third, and second and fourth factors.]

6. $x(x+1)^2(x+2)=72$. (D. U. 1944)

7. $(x+2)(x+3)(x+5)(x+6)=504$ (H. S. 1947, Q. 1950)

8. $16x(x+1)(x+2)(x+3)=9$. (H. S. 1949)

1.54. Fourth Type. Reciprocal Equations. If an equation in x is unaltered by changing x into $\frac{1}{x}$, it is known as a reciprocal equation. In such an equation the co-efficients of terms equidistant from the beginning and end are equal and of the same or different signs ; for example :

$$ax^3+bx^2+bx+a=0 \quad \dots(1)$$

$$ax^3-bx^2+bx-a=0 \quad \dots(2)$$

$$ax^4+bx^3+cx^2+bx+a=0 \quad \dots(3)$$

$$ax^4-bx^3+bx-a=0, \quad \dots(4)$$

are reciprocal equations. Evidently $ax^4-bx^3+cx^2+bx-a=0$ is not a reciprocal equation, but we shall see that the method of solution is the same in this case also.

Important Conclusions :

(i) When an equation is of even degree and co-efficients from the beginning and end differ only in sign, as in (4) above, the middle term must be absent for a reciprocal equation.

(ii) By actual substitution it is clear that $x=-1$ is a root of (1), $x=1$ is a root of (2) and $x=\pm 1$ are roots of (4). We shall make use of this fact in solving equations of this type. We now illustrate the method of solving these equations.

Ex. 1. Solve : $2x^3-3x^2-3x+2=0$.

$x=-1$ is obviously a root. Hence arranging, we get

$$2(x^3+1)-3x(x+1)=0$$

or $(x+1)[2(x^2-x+1)-3x]=0$

or $(x+1)(2x^2-5x+2)=0$

$\therefore (x+1)=0$, i. e., $x=-1$ as already known,

and $2x^2-5x+2=0$, i. e. $x=\frac{5\pm\sqrt{25-16}}{4}=2$, or $\frac{1}{2}$.

The following example illustrates the method of solving equations of the type (3), an equation of an even degree in which the middle term is not absent.

Ex. 2 Solve : $2x^4-x^3-11x^2-x+2=0$.

Dividing by x^2 , we get

$$2x^2-x-11-\frac{1}{x}+\frac{2}{x^2}=0$$

$$\text{or } 2\left(x^2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)-11=0.$$

Now putting $x+\frac{1}{x}=z$, we have $x^2+\frac{1}{x^2}+2=z^2$.

and $\therefore x^2+\frac{1}{x^2}=z^2-2$, and the equation becomes

$$2(z^2-2)-z-11=0, \text{ or } 2z^2-z-15=0,$$

$$\therefore z=\frac{1\pm\sqrt{1+120}}{4}=3 \text{ or } \frac{-5}{2},$$

$$\text{i. e., } x+\frac{1}{x}=3$$

$$\text{or } x+\frac{1}{x}=\frac{-5}{2}.$$

$$\text{i. e., } x^2-3x+1=0$$

$$\text{or } 2x^2+5x+2=0$$

$$\text{i. e., } x=\frac{3\pm\sqrt{9-4}}{2}$$

$$\text{or } x=\frac{-5\pm\sqrt{25-16}}{4}$$

$$\text{i. e., } x=\frac{3\pm\sqrt{5}}{2}$$

$$\text{or } x=-2 \text{ or } \frac{1}{2}.$$

Working Rule :

To solve reciprocal equations the following steps will prove useful :

(i) If equation is of odd degree 1 or -1 will be a root. Cancel the factor and remaining will be even degree.

(ii) For even degree reciprocal equation see whether 1 and -1 are roots if not divide by the middle term without coefficient ; if so cancel out x^2-1 and then divide by middle term as above.

(iii) Do not forget to write roots corresponding to cancelled factors.

EXERCISE VI

Solve :

1. $2x^3 - 7x^2 + 7x - 2 = 0.$

2. $x^3 + 6x^2 + 6x + 1 = 0.$

3. $2x^3 - 3x^2 - 3x + 2 = 0.$

4. $x^4 + 3x^3 - 3x - 1 = 0.$

5. $2x^4 + 9x^3 + 14x^2 + 9x + 2 = 0.$

6. $2x^5 + x^4 + x + 2 = 12x^3 + 12x^2.$

7. $6 \left(x^2 + \frac{1}{x^2} \right) + 5 \left(x + \frac{1}{x} \right) - 38 = 0.$

1.55. Fifth Type : Sometimes equations may be given involving square roots of quadratic expressions. Such equations can be easily solved when one linear factor happens to be common throughout as in the following example. Care however should be taken to see that the root corresponding to the cancelled out factor is also to be written.

Ex. Solve : $\sqrt{x^2 + 4x - 21} + \sqrt{x^2 - x - 6} = \sqrt{6x^2 - 5x - 39}.$

When expressions under the radicals are factorised we get

$$\sqrt{(x-3)(x+7)} + \sqrt{(x-3)(x+2)} = \sqrt{(x-3)(6x+13)}$$

$$\therefore \sqrt{x-3} = 0 \text{ or } x=3$$

and $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}.$

This is already solved as Ex. 2 Art. 1.31 page 8.

EXERCISE VII

Solve the following equations :

1. $\sqrt{x^2 + 4x - 5} + \sqrt{x^2 + 5x - 6} = \sqrt{2x^2 + 9x - 11}.$

2. $x + 2 + \sqrt{x^2 - x - 6} = 5\sqrt{x + 2}.$

3. $\sqrt{2x^2 + 9x + 7} + \sqrt{3(x^2 - 5x - 6)} = \sqrt{7x^2 + 8x + 1}.$

4. $\sqrt{a^2 + 2ax - 3x^2} - \sqrt{a^2 + ax - 6x^2} = \sqrt{2a^2 + 3ax - 9x^2}.$

5. $\sqrt{x^2 - 16} - (x - 4) = \sqrt{x^2 - 5x + 4}. \quad (\text{D. U. 1949}).$

6. $\sqrt{x^2 + 6x - 16} + \sqrt{x^2 + x - 6} = \sqrt{x^2 + 2x - 8}. \quad (\text{D. U. 1948}).$

7. $\sqrt{x^2 - 16} - \sqrt{x - 4} = \sqrt{x^2 - 5x + 4}. \quad (\text{D. U. 1944}).$

1.56. Solutions by Inspection :

(a) Some of the equations although not quadratics can be solved when a root is known by trial,

e.g. Solve $x^3 - 4x^2 + 4x - 1 = 0.$

Here $x=1$ satisfies the given equation and after cancelling the factor $x-1$ from L. H. S. we get

$$x^2 + x + 1 - 4x = 0.$$

$$\text{or } x^2 - 3x + 1 = 0.$$

$$x = \frac{3 \pm \sqrt{9-4}}{2}$$

$\therefore$ roots of equation are

$$\frac{3 \pm \sqrt{5}}{2} \text{ and } 1.$$

(b) Sometimes consideration of an identity helps to solve equations :—

$$\text{Solve : } \sqrt{2x^2 - 3x - 5} + \sqrt{x^2 - 3x + 4} = x - 3. \quad \dots(i)$$

Identically we have

$$(2x^2 - 3x - 5) - (x^2 - 3x + 4) = x^2 - 9 \quad \dots(ii)$$

Dividing (ii) by (i) we have

$$\sqrt{2x^2 - 3x - 5} - \sqrt{x^2 - 3x + 4} = x + 3 \quad \dots(iii)$$

Adding (i) and (iii) and dividing by 2 we have

$$\sqrt{2x^2 - 3x - 5} = x$$

$$\text{or } 2x^2 - 3x - 5 = x^2$$

$$\text{or } x^2 - 3x - 5 = 0$$

$$\therefore x = \frac{3 \pm \sqrt{29}}{2}$$

EXERCISE VIII

Solve the following equations :

1. $2x^3 - 7x^2 + 7x - 2 = 0.$

2. $x^3 - 5x^2 + x + 10 = 0.$

3. $x^3 + x^2 - x - 1 = 0.$

4. $2x^3 + 3x^2 - 2x - 3 = 0.$

5. $\sqrt{2x^2 - 4x + 9} - \sqrt{2x^2 - 6x + 4} = 3.$

6. $\sqrt{5x^2 + 7x + 2} - \sqrt{4x^2 + 7x + 18} = x - 4.$

7. $\sqrt{x^2 - a + b} + \sqrt{x^2 - ax + c} = \sqrt{b} + \sqrt{c}.$

8. $\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1.$

(P. U. 1950)

CHAPTER II

SIMULTANEOUS QUADRATIC EQUATIONS

No general rules can be laid down for the solution of quadratic equations involving two or more variables. The following articles will illustrate the various artifices employed for the solution of such equations.

Equations involving two variables.

2.1. By expressing one variable in terms of the other.

Ex. 1. Solve

$$x + xy = 9 \quad \dots(1)$$

$$y + xy = 8. \quad \dots(2)$$

Subtracting (2) from (1), we have

$$x - y = 1 \quad \text{or } x = 1 + y \quad \dots(3)$$

Substituting this in (1), we get

$$1 + y + (1 + y)y = 9$$

$$\text{or } y^2 + 2y - 8 = 0 \quad \text{or } (y + 4)(y - 2) = 0$$

$$\text{which gives } y = 2 \quad \text{or } -4$$

$$\text{and from (3) we get } x = 3 \quad \text{or } -3$$

Hence the roots are

$$x = 3 \quad y = 2 \quad \text{or } x = -3 \quad y = -4.$$

$$\textbf{Ex. 2.} \quad \text{Solve } 6x^2 - 5xy + y^2 = 0 \quad \dots(1)$$

$$3x^2 + 2y + y^2 = 60. \quad \dots(2)$$

Dividing (1) by x^2 , we get

$$\frac{y^2}{x^2} - 5 \frac{y}{x} + 6 = 0$$

$$\text{which gives } \left(\frac{y}{x} - 3 \right) \left(\frac{y}{x} - 2 \right) = 0$$

$$\text{i.e., } y = 3x \quad \dots(3)$$

$$\text{or } y = 2x \quad \dots(4)$$

From (2) and (3), we get

$$3x^2 + 6x + 9x^2 = 60, \quad \text{or } 2x^2 + x - 10 = 0$$

$$\text{whence } x = -2 \quad \text{or } -\frac{5}{2}.$$

$\therefore$ from (3) $y=6$ or $-\frac{15}{2}$.

Again from (2) and (4), we have,

$$7x^2 + 4x - 60 = 0$$

whence $x = \frac{-2 \pm 2\sqrt{106}}{7}$

and from (4) $y = \frac{-4 \pm 4\sqrt{106}}{7}$.

EXERCISE IX

Solve the following equations :

1. $3x - 2y = 4$
 $x^2 - 3xy = -2.$

3. $x + xy = 3$
 $y + xy = 4$

5. $3x + 4y = 25$
 $\frac{3}{x} + \frac{4}{y} = 2.$

7. $x^2 + 8xy - y^2 = 16$
 $x + 3y = 10.$

9. $x^2 + xy = 15$
 $x^2 - y^2 = 5.$

11. $x^2 - y^2 = 1$
 $x^4 - 2y^4 = 1. \quad (\text{D. Q. 1951})$

13. $x^2 + y^2 = \frac{5}{16}$
 $x - y = \frac{1}{4}.$

2. $2y - x + 1 = 0$
 $\frac{3}{x} + \frac{1}{y} - 2 = 0.$

4. $(x-a)(y-b) = ab$
 $\frac{x}{a} = \frac{y}{b}.$

6. $3x - 4y = 5$
 $2x^2 - 5xy - 3y^2 = 0$

8. $x(a-x) + y(b-y) = 0$
 $x - y = a - b.$

10. $\frac{57}{x+y} + \frac{6}{x-y} = 5$
 $\frac{38}{x+y} + \frac{21}{x-y} = 9.$

(H. S. 1946).

12. $ax^2 + by^2 = a + b$
 $x + y = 1. \quad (\text{K. U. 1950})$

(K. U. 1951)

2.2. By obtaining the values of the sum and difference of the two variables:

Ex. 1. Solve $x^2 + y^2 = 53$
 $xy = 14$

...(1)

...(2)

Multiply (2) by 2 and then by addition and subtraction, we get

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$$\begin{array}{ll} (x+y)^2=81 & \text{or } x+y=\pm 9. \\ \text{or } (x-y)^2=25 & \text{or } x-y=\pm 5. \end{array}$$

These results furnish *four* pairs of simple equations, viz.,

$$\begin{array}{llll} x+y=9 & x+y=9 & x+y=-9 & x+y=-9 \\ x-y=5, & x-y=-5, & x-y=5, & x-y=-5 \end{array}$$

By addition and subtraction, we get

$$\begin{array}{llll} x=7 & x=2 & x=-2 & x=-7 \\ y=2 & y=7 & y=-7 & y=-2. \end{array}$$

So that there are *four* solutions.

$$\text{Ex. 2. Solve } \frac{a}{x} + \frac{b}{y} = 2 \quad \dots(1)$$

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 2. \quad \dots(2)$$

By subtracting (2) from the square of (1), we get

$$\frac{2ab}{xy} = 2. \quad \dots(3)$$

From (2) and (3), we have

$$\left(\frac{a}{x} - \frac{b}{y} \right)^2 = 2 - 2 \text{ i.e., } \frac{a}{x} - \frac{b}{y} = 0. \quad \dots(4)$$

From (1) and (4) by addition and subtraction we have
 $x=a$ and $y=b$.

2.3. Some equations of degree higher than the second can also be solved by the above method.

$$\text{Ex. 1. } x^3 + y^3 = 351 \quad \dots(1)$$

$$x + y = 9. \quad \dots(2)$$

Dividing (1) by (2), we get

$$\frac{x^3 + y^3}{x + y} = 39, \text{ i.e., } x^2 - xy + y^2 = 39. \quad \dots(3)$$

Subtracting (3) from the square of (2), we get

$$3xy = 42, \text{ i.e., } xy = 14 \quad \dots(4)$$

Now from (3) and (4), we have

$$(x - y)^2 = 39 - 14 = 25 \text{ or } x - y = \pm 5 \quad \dots(5)$$

From (2) and (5) by addition and subtraction, we have

$$\begin{array}{ll} x=7 & \text{or } x=2 \\ y=2 & \text{or } y=7. \end{array}$$

Ex. 2. Solve $x^3 + y^3 = 35$... (1)
 $x^2y + xy^2 = 30$ (2)

Adding 3 times (2) to (1), we get

$$(x+y)^3 = 125$$

or $x+y=5$.

From (2), we have ... (3)

$$xy(x+y) = 30$$

$$xy = 6.$$

From (3) and (4), $(x-y)^2 = 25 - 24$ i.e., $x-y = \pm 1$... (4)

From (3) and (5), we get ... (5)

$$x=3 \quad y=2 \quad \text{or} \quad x=2 \quad y=3.$$

Ex. 3. Solve $xy(x+y) = 30$, $x^3 + y^3 = 35$.

Here dividing we get

$$\frac{x^2 - xy + y^2}{xy} = \frac{35}{30} = \frac{7}{6}$$

$$\therefore 6x^2 - 13xy + 6y^2 = 0 \text{ i.e., } (2x-3y)(3x-2y) = 0$$

$$\therefore 2x=3y, \text{ or } 3x=2y$$

Now it can be easily solved.

EXERCISE X

Solve the following equations :

1. $x^2 + y^2 = 13$
 $xy = 6$.

2. $2x^2 + 2y^2 = 5$
 $7x^2 - 10xy + 7y^2 = 10$.

3. (a) $x+y=5$, (b) $x-y=3$, $x^2+y^2=185$. (D. U.)
 $x^2+y^2=8xy$.

4. If $z=x-y$, obtain from the equations
 $xy=y-x+13$ and $x^2+y^2=25$,
 a quadratic equation in z , and thence find the values of x and
 y satisfying the equations.

5. $x^3 + y^3 = 91$
 $x+y=7$.

6. $x-y=1$
 $x^3 - y^3 = 19$.
 (P. U. 1950)

7. $x^3 - y^3 = 218$.
 $x-y=2$.

8. $x+y=16$
 $x^3 + y^3 = 1072$.
 (Q. 1950).

9. $x^3 + y^3 = \frac{9}{2}xy$ 10. $\frac{x^2}{y} + \frac{y^2}{x} = 28$
 $x+y=3$. $x+y=12$.

11. $xy(x+y)=30$
 $x^3+y^3=35$.

12. $x^3 - y^3 = 124$
 $x^2y - xy^2 = 20$.

$$13. \quad bx + ay = a^2 + b^2$$

$$14. \quad \frac{1}{x} + \frac{1}{y} = 5.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{b^2}{a^2} + \frac{a^2}{b^2}.$$

$$\frac{1}{x^2} + \frac{1}{y^2} = 13.$$

$$15. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$16. \quad \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{10}{3}$$

$$\frac{x}{a} - \frac{y}{b} = 1$$

$$x + y = 10.$$

$$17. \quad \frac{x}{2} + \frac{y}{5} = 5, \quad \frac{2}{x} + \frac{5}{y} = \frac{5}{6}.$$

(D.U. 1943)

2.4. By substituting $y=mx$ in equations involving homogeneous expressions in x and y .

Ex. Solve : $5y^2 - 7x^2 = 17$
 $5xy - 6x^2 = 6.$

Putting $y=mx$ in both the equations we get,

$$x^2(5m^2 - 7) = 17 \quad (1)$$

$$\text{and } x^2(5m - 6) = 6. \quad (2)$$

$$\therefore \frac{5m^2 - 7}{5m - 6} = \frac{17}{6} \quad \text{or } 30m^2 - 42 = 85m - 102$$

$$\text{i.e., } 6m^2 - 17m + 12 = 0$$

$$\therefore m = \frac{3}{2} \text{ or } \frac{4}{3}.$$

Substituting $m = \frac{3}{2}$ in (2), we get $x^2 = 4$, or $x = \pm 2$
 and $y = mx = \frac{3}{2}x = \pm 3.$

Again substituting $m = \frac{4}{3}$ in (2), we have

$$x^2 = 9 \text{ or } x = \pm 3,$$

$$\text{and } y = mx = \frac{4}{3}x = \pm 4.$$

Hence $x = 2, 3, -2, -3.$

$$y = 3, 4, -3, -4.$$

EXERCISE XI

Solve :

$$1. \quad x^2 + xy = 77$$

$$xy - y^2 = 12.$$

$$3. \quad x^2 + y^2 = 1$$

$$x^2 + 15xy - 4y^2 = 6.$$

$$2. \quad x^2 - xy + y^2 = 19$$

$$x^2 + xy + y^2 = 49.$$

$$4. \quad 3x^3 - 8xy^2 + y^3 + 21 = 0$$

$$x^2(y - x) = 1.$$

$$5. \quad x^2 + y^2 = a^2, \quad x + y = b. \qquad 6. \quad x^2 - 3xy + 2y^2 = 24$$

$$7. \quad \begin{aligned} x + y &= 8xy, \\ x^2 + y^2 &= 40x^2y^2. \end{aligned} \qquad x^2 + 3xy - y^2 = 9.$$

(D. U.)

8. The area of a rectangular floor is 297 sq. ft. If it had been 6 inches longer and also 6 inches narrower, the area would have been 1 sq. ft. less. Find its length and breadth.

2. 5. Equations involving three variables.

Ex. 1. Solve :

$$x + 2y + 5z = 0 \qquad \dots(1)$$

$$2x + 3y + 4z = 0 \qquad \dots(2)$$

$$x^2 + xy + z^2 = 72. \qquad \dots(3)$$

From (1) and (2), by cross-multiplication, we get

$$\frac{x}{8-15} = \frac{y}{10-4} = \frac{z}{3-4} \quad \text{or} \quad \frac{x}{-7} = \frac{y}{6} = \frac{z}{-1}$$

$$\text{or} \quad x = 7z \text{ and } y = -6z. \qquad \dots(4)$$

Substituting these in (3), we have $49z^2 - 42z^2 + z^2 = 72$.

$$\text{or} \quad z^2 = 9. \qquad \text{i.e., } z = \pm 3,$$

$\therefore$ from (4),

$$\text{when } z = 3,$$

$$x = 21,$$

$$y = -18,$$

$$\text{or} \quad \text{when } z = -3.$$

$$x = -21,$$

$$y = 18.$$

Ex. 2. Solve :

$$yz = 2 \qquad \dots(1)$$

$$zx = 3 \qquad \dots(2)$$

$$xy = 6. \qquad \dots(3)$$

Multiplying (1), (2) and (3), we get

$$x^2y^2z^2 = 36 \qquad \text{or } xyz = \pm 6. \qquad \dots(4)$$

Dividing (4) by (1), (2) and (3) respectively, we get

$$x = \pm 3, \quad y = \pm 2, \quad z = \pm 1.$$

Since these values of x, y, z have to satisfy equations (1), (2) and (3) we have

$$x = 3, \quad y = 2, \quad \text{and } z = 1.$$

$$\text{or} \quad x = -3, \quad y = -2 \quad \text{and } z = -1.$$

Ex. 3. Solve :

$$x(x + y + z) = 18. \qquad \dots(1)$$

$$y(x + y + z) = 40. \qquad \dots(2)$$

$$z(x + y + z) = 63. \qquad \dots(3)$$

Adding (1), (2) and (3), we get

$$(x+y+z)^2=121$$

$$x+y+z=\pm 11$$

Dividing (1), (2) and (3) by (4), we have

$$x=\pm \frac{1}{11}, y=\pm \frac{4}{11}, z=\pm \frac{6}{11},$$

Hence $x=\frac{1}{11}$, $y=\frac{4}{11}$ and $z=\frac{6}{11}$.

or $x=-\frac{1}{11}$, $y=-\frac{4}{11}$ and $z=-\frac{6}{11}$.

EXERCISE XII

Solve :

$$\begin{aligned} 1. \quad & 9x+y-8z=0 \\ & 4x-8y+7z=0 \\ & yz+zx+xy=47. \end{aligned}$$

$$\begin{aligned} 2. \quad & 2x+y+z=1 \\ & 3x+2y+z=4 \\ & x^2+y^2+z^2+2yz=2. \end{aligned}$$

$$\begin{aligned} 3. \quad & yz+zx=16 \\ & zx+xy=25 \\ & xy+yz=-39. \end{aligned}$$

$$\begin{aligned} 4. \quad & x(y+z)=100 \\ & y(z+x)=144 \\ & z(x+y)=154. \end{aligned}$$

$$\begin{aligned} 5. \quad & x^2yz=a^4 \\ & xy^2z=b^4 \\ & xyz^2=c^4. \end{aligned}$$

$$\begin{aligned} 6. \quad & x^2+xy+xz=48 \\ & xy+y^2+yz=12 \\ & xz+yz+z^2=84. \end{aligned}$$

$$\begin{aligned} 7. \quad & (x+y)(x+z)=56 \\ & (y+z)(y+x)=63 \\ & (z+x)(z+y)=72. \end{aligned}$$

$$\begin{aligned} 8. \quad & x+y-z=14 \\ & y^2+z^2-x^2=46 \\ & yz=9. \end{aligned}$$

$$\begin{aligned} 9. \quad & xy+x+y=23 \\ & zx+x+z=41 \\ & yz+y+z=27. \end{aligned}$$

$$10. \quad \frac{xy}{x+y}=1, \quad \frac{yz}{y+z}=2, \quad \frac{zx}{x+z}=3.$$

$$\begin{aligned} 11. \quad & xz+y=7z \\ & yz+x=8z \\ & x+y+z=12. \end{aligned}$$

$$\begin{aligned} 12. \quad & y+z : z+x : x+y = a : b : c \\ & (y+z)^2 + (z+x)^2 + (x+y)^2 = 1. \end{aligned}$$

$$13. \quad \text{Solve } 8x+2y-9z=0, \quad 3x+y-3z=0$$

$$\frac{3}{5}x - \frac{2}{5}y + \frac{1}{5}z = 10$$

and find the value of $x^2+y^2+z^2$.

26. Problems involving the solution of simultaneous equations.

Ex. Find two numbers, the sum of whose squares is equal to 17 times their sum and also to 68 times their difference.

Let x and y be the two numbers.

$$\text{Then } x^2 + y^2 = 17(x + y) = 68(x - y)$$

$$\therefore x + y = 4(x - y) \text{ i.e., } y = \frac{3}{5}x.$$

$$\therefore \text{Substituting we have } x^2 + \frac{9}{25}x^2 = 17(x + \frac{3}{5}x)$$

$$\therefore 34x^2 = 17 \times 8 \times 5x \text{ so that } x = 20 \text{ and } \therefore y = \frac{3}{5}x = 12.$$

EXERCISE XIII

1. A man rows to a place $16\frac{1}{2}$ miles down a river and back in 6 hours 40 minutes. If the rate of the stream had been twice as great, he would have taken 6 hours 52 $\frac{1}{2}$ minutes. Find the rates of the stream and the man.

2. Find two numbers the sum of whose squares is equal to 25 times their sum and also to 50 times their difference.

3. Two numbers, each of two digits, are composed of the same two digits in the reverse order. The difference of the squares of the two numbers is 891 and the greater is 6 times the sum of its digits. Find the numbers.

4. A cyclist rides from A to B and back. A pedestrian starts at the same time from A to walk to B, but turns back owing to rain when he has gone one-third of the distance, and is passed by the cyclist five miles from A. If the cyclist rides 11 miles while the pedestrian walks 3, what is the distance from A to B?

CHAPTER III

THEORY OF QUADRATIC EQUATIONS

3.1. To show that a quadratic equation cannot have more than two different roots.

If possible, let the quadratic equation $ax^2 + bx + c = 0$, have three different roots α , β and γ .

Then from the definition of a root, we have

$$a\alpha^2 + b\alpha + c = 0 \quad \dots(1)$$

$$a\beta^2 + b\beta + c = 0 \quad \dots(2)$$

and $a\gamma^2 + b\gamma + c = 0 \quad \dots(3)$

Subtracting (2) from (1), we have

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0, \text{ or } (\alpha - \beta)[a(\alpha + \beta) + b] = 0.$$

But $\alpha - \beta$ is not equal to zero, because α and β are different

$$\therefore a(\alpha + \beta) + b = 0 \quad \dots(4)$$

Similarly from (2) and (3) we get

$$a(\beta + \gamma) + b = 0 \quad \dots(5)$$

Subtracting (5) from (4), we get $a(\alpha - \gamma) = 0$

which is impossible, since neither $a = 0$, nor α is equal to γ (6)

Hence a quadratic equation cannot have more than two different roots. (For another proof see Art. 3.4)

3.2. An **identity** is a statement of equality between two expressions which is true for all values of the letter or letters concerned.

For example,

$$(x+a)(x+b) \equiv x^2 + x(a+b) + ab,$$

and $(x+a)^2 \equiv x^2 + 2ax + a^2$ are identities.

The sign $\equiv$ being used to distinguish it from the sign of equality.

3.21. To show that when an equation in any variable is true for many values of the variable it is an identity.

Let us suppose $ax^2 + bx + c = 0$ be satisfied by the values α , β , γ , (more than two) of x .

Then by (6) Art. 3.1 we see that $a = 0$ then by (4) $b = 0$ and again by (1) $c = 0$

$$\therefore ax^2 + bx + c \equiv 0$$

EXERCISE XIV

1. Show that :

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1$$

is an identity.

[Hint : Show that it holds true for $x=a$, b and c].

2. Show that

$$(x-2)(x-3) - 8(x-3)(x-1) + 9(x-1)(x-2) = 2x^2$$

is an identity.

3. If $(p-a)^3 + (p-b)^3 = 2(p-c)^3$

$$(q-a)^3 + (q-b)^3 = 2(q-c)^3$$

and $(r-a)^3 + (r-b)^3 = 2(r-c)^3$

show that

$$(s-a)^3 + (s-b)^3 = 2(s-c)^3, \text{ where } s \text{ is different from } p, q \text{ and } r.$$

[Hint : Consider the quadratic equation $(x-a)^3 + (x-b)^3 = 2(x-c)^3$ and show that it is an identity.]

Relation between the Roots and the Co-efficients.

3.3. To find the relation between the roots and the co-efficients of a quadratic equation.

Let α, β be the roots of the equation $ax^2 + bx + c = 0$.

Then we know that

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \alpha + \beta = -\frac{b}{a}$$

i. e. **sum of roots** = $\frac{\text{co-efficient of } x \text{ with sign changed}}{\text{co-efficient } x^2}$

$$\begin{aligned} \alpha\beta &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} \end{aligned}$$

i. e. **product of roots** = $\frac{\text{constant term}}{\text{co-efficient of } x^2}$

3.31. If α, β be the roots of $ax^2 + bx + c = 0$, then
 $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.

For $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$
 $= a[x^2 - (\alpha + \beta)x + \alpha\beta]$
 $= a(x - \alpha)(x - \beta).$

Ex. 1. If α, β be the roots of the equation $ax^2 + bx + c = 0$, find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$ (iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (H. S. B. Delhi 1950)

Here we know that

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = -\frac{c}{a}$$

$$\begin{aligned} \therefore (i) \quad \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{b}{a}\right)^2 - 2\left(-\frac{c}{a}\right) \\ &= \frac{b^2 - 2ac}{a^2} \end{aligned}$$

$$(ii) \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-b}{a} \div \frac{-c}{a} = \frac{-b}{c}.$$

$$\begin{aligned} (iii) \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} \\ &= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} \\ &= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}} = \frac{b^2 - 2ac}{c^2}. \end{aligned}$$

Ex. 2. If one root of $2x^2 + px + 3$, be 3, find the other root and the value of p .

Here let β be the other root.

Then $\beta + 3 = -\frac{p}{2}$... (i)

and $3\beta = \frac{3}{2}$ i. e. $\beta = \frac{1}{2}$

From (i) $\frac{1}{2} + 3 = -\frac{p}{2}$ or $1 + 6 = -p$ i. e. $p = -7$.

EXERCISE XV-A

1. If α, β be the roots of $x^3 - px + q = 0$; evaluate
 - (i) $\alpha^3 + \beta^3$
 - (ii) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$
 - (iii) $\alpha^2 + \beta^2 + \frac{\alpha\beta}{\alpha + \beta}$
 - (iii) $\alpha^4 + \beta^4$.
2. If one root of $3x^2 + 9x + q = 0$ be -1 . find the value of q and the other root.
3. If α, β be the roots of $x^2 - 4x + 1 = 0$, evaluate

$$\frac{2-\alpha}{4+\alpha} + \frac{2-\beta}{4+\beta}$$
4. If α, β are the roots of $ax^2 + bx + c = 0$, find the value of
 - (i) $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2}$
 - (ii) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$ (K. U. 1951)
5. If α, β are the roots of $x^2 - qx + r = 0$, find the value of $(a\alpha^2 + b)(a\beta^2 + b)$.
6. If α, β be the roots of $px^2 - qx + r = 0$, find the value of
 - (i) $\sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{\alpha}{\beta}}$
 - (ii) $\frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$.
7. If α, β are the roots of the equation $x^2 + px + q = 0$, find the terms of p and q the value of
 - (i) $\frac{(p+\alpha)(p+\beta)}{(q-\alpha^2)(q-\beta^2)}$
 - (ii) $\alpha^3\beta + \beta^3\alpha$. (K. U. 1949)
8. If α, β , be the roots of $x^2 + px + q = 0$ and α', β' those of $x^2 - p'x + q' = 0$, find the value of $(\alpha - \alpha')(\beta - \beta') + (\alpha - \beta')(\beta - \alpha')$.
9. Find the values of p if one root of the equation $27x^2 + 6x - (p+2) = 0$, is the square of the other.
10. If one of the roots of $3x^2 + 5x + a = 0$ is four times the other; find the value of a and the roots.
11. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, prove that $b^2 - 4c = 1$. (Q. 1951)
12. Find the condition that the roots of the equation $x^2 - bx + c = 0$ may differ by $3c$.

3.4. A quadratic equation cannot have more than two roots.

Let $ax^2+bx+c=0$ (1)
be the general quadratic equation and let α, β be its roots,
then we know $\alpha+\beta=-\frac{b}{a}$

$$\alpha\beta=\frac{c}{a}$$

$$\begin{aligned}\therefore \text{ we have } ax^2+bx+c &= a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] \\ &= a[x^2-(\alpha+\beta)x+\alpha\beta] \\ &= a(x-\alpha)(x-\beta). \quad \dots (2)\end{aligned}$$

This shows that $x-\alpha, x-\beta$ are the factors of the expression ax^2+bx+c .

So that the roots of equation (1) are α, β obtained by equating (2) to zero.

The expression ax^2+bx+c being only of the second degree it cannot have any other factor involving x .

Hence the equation (1) cannot have any other root. For if γ be a possible third root of (1) we shall have

$$a\gamma^2+b\gamma+c=0.$$

Also from (2), we have

$$a\gamma^2+b\gamma+c=a(\gamma-\alpha)(\gamma-\beta).$$

$$\therefore a(\gamma-\alpha)(\gamma-\beta)=0.$$

Which is absurd as none of the factors $\gamma-\alpha, \gamma-\beta$ is zero.

3.41. In Art. 1.3 it was shown that a quadratic equation has two roots. This combined with Art 3.4 leads to the conclusion that: A quadratic equation has two and only two roots. (cf. also 3.1)

3. 42. If α be a root of $ax^2+bx+c=0$ then to show that $x-\alpha$ is a factor of ax^2+bx+c and conversely (Factor Theorem).

Here if α be a root of

$$ax^2+bx+c=0 \text{ then}$$

$$a\alpha^2+b\alpha+c=0 \dots\dots\dots(i)$$

Let ax^2+bx+c when divided by $x-\alpha$ leave R as remainder and give Q as quotient, then

$$ax^2+bx+c \equiv Q(x-\alpha)+R.$$

Putting $x=a$ on both sides and using (i) we have

$$0=Q \times 0 + R$$

$$\therefore R=0.$$

Conversely :— If $x-a$ be a factor of ax^2+bx+c then
 $ax^2+bx+c=A(x-a)$

Putting $x=a$ we see $a^2+ba+c=0$ which proves the theorem.

Cor. If α, β be roots of $ax^2+bx+c=0$ then

$$ax^2+bx+c \equiv a(x-\alpha)(x-\beta).$$

3.5. To form an equation whose roots are α, β .

Let the quadratic equation whose roots are α, β be
 $ax^2+bx+c=0$.

$\therefore$ The equation can be written, after division by a as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

$$\text{or } x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$\text{i.e., } x^2 - (\text{sum of roots})x + \text{product of roots} = 0.$$

Note — The same follows from the obvious fact that the equation whose roots are α, β is $(x-\alpha)(x-\beta)=0$

$$\text{i.e., } x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

Ex. 1. Form an equation whose roots are $-3, 4$.
 Here the required equation is

$$x^2 - (\text{sum of roots})x + \text{product} = 0$$

$$\therefore x^2 - (4-3)x + 4(-3) = 0$$

$$x^2 - x - 12 = 0.$$

Otherwise thus :—

Required equation is

$$[x - (-3)][x - 4] = 0$$

$$\text{or } x^2 - x - 12 = 0.$$

Ex. 2 If one root of an equation be $p + \sqrt{q}$; find the equation and the other root.

$$\text{Here } x = p + \sqrt{q}$$

$$\therefore (x-p)^2 = q$$

$$\therefore x^2 - 2px + p^2 - q = 0$$

The other root is $p - \sqrt{q}$.

Ex. 3. Find an equation whose roots are squares of the roots of $ax^2+bx+c=0$.
 (D. U.)

Here if α, β are the roots of $ax^2 + bx + c = 0$

$$\text{then } \alpha + \beta = \frac{-b}{a} \quad \dots (1)$$

$$\alpha\beta = \frac{c}{a} \quad \dots (2)$$

The required equation having α^2, β^2 as roots is

$$\begin{aligned} & (x - \alpha^2)(x - \beta^2) = 0 \\ \text{or } & x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0 \\ \text{i.e. } & x^2 - \{(\alpha + \beta)^2 - 2\alpha\beta\}x + (\alpha\beta)^2 = 0 \\ \therefore & \text{ using (1) and (2),} \end{aligned}$$

$$x^2 - \left(\frac{b^2}{a^2} - \frac{2c}{a}\right)x + \frac{c^2}{a^2} = 0$$

$$\begin{aligned} \text{i.e. } & a^2x^2 - (b^2 - 2ac)x + c^2 = 0. \\ \text{or } & a^2x^2 + (2ac - b^2)x + c^2 = 0. \end{aligned}$$

Otherwise thus :—The given equation is

$$\begin{aligned} & ax^2 + c = -bx \\ \therefore & (ax^2 + c)^2 = b^2x^2 \\ \text{or } & a^2x^4 + c^2 + 2acx^2 = b^2x^2 \\ \text{i.e., } & a^2x^4 + x^2(2ac - b^2) + c^2 = 0. \end{aligned}$$

Now let $y = x^2$

$$\therefore a^2y^2 + y(2ac - b^2) + c^2 = 0. \quad (i)$$

Now when x has values α, β ; y will have α^2, β^2 .

$\therefore (i)$ is the required equation having α^2, β^2 as roots.

Note.—The second method is quite general and can be followed for the third and higher powers of the roots also.

Ex. 4. If α, β be the roots of $ax^2 + bx + c = 0$ form an equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

We know that $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$.

The sum of the roots of the required equation is given by

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \frac{\frac{b^2}{a^2} - 2\frac{c}{a}}{\frac{c^2}{a^2}} = \frac{b^2 - 2ac}{c^2}.$$

$$\text{Product of roots} = \frac{1}{\alpha^2} \times \frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2} = \frac{1}{\frac{c^2}{a^2}} = \frac{a^2}{c^2}.$$

Now any quadratic equation can be written as

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

$\therefore$ the required equation is

$$x^2 - \left(\frac{b^2 - 2ac}{c^2} \right) x + \frac{a^2}{c^2} = 0,$$

$$\text{or } c^2x^2 - (b^2 - 2ac)x + a^2 = 0.$$

Otherwise thus :—

Let $y = \frac{1}{x^2}$ and then proceeding as in Ex. 3 above we can get the equation.

Ex. 5. Find the condition that one root of $px^2 + qx + r = 0$, may be double of the other.

Let one root be α and the other be β , so that $\beta = 2\alpha$.

$$\text{We have } \alpha + 2\alpha = -\frac{q}{p} \text{ i.e., } 3\alpha = -\frac{q}{p} \quad \dots\dots(i)$$

$$\text{and } \alpha \cdot 2\alpha = \frac{r}{p} \quad \text{i.e., } 2\alpha^2 = \frac{r}{p}. \quad \dots\dots(ii)$$

Substituting the value of α from (i) in (ii), we get

$$2\left(-\frac{q}{3p}\right)^2 = \frac{r}{p} \text{ or } \frac{2q^2}{9p^2} = \frac{r}{p}, \text{ i.e., } 2q^2 = 9pr.$$

Ex. 6. Find the condition that one root of $ax^2 - bx + c = 0$ may be square of the other.

Let one of the roots be α and the other β so that $\alpha^2 = \beta$.

$$\text{Now } \alpha + \alpha^2 = \frac{b}{a} \quad \dots\dots(1)$$

$$\text{and } \alpha \cdot \alpha^2 = \frac{c}{a}, \text{ or } \alpha^3 = \frac{c}{a}, \text{ i.e., } \alpha = \left(\frac{c}{a}\right)^{\frac{1}{3}}.$$

Substituting this value of α in (i), we get

$$\left(\frac{c}{a}\right)^{\frac{1}{3}} + \left(\frac{c}{a}\right)^{\frac{2}{3}} = \frac{b}{a}. \quad \dots\dots(2)$$

Cubing both the sides and using (2), we can put this as

$$\frac{c}{a} + \frac{c^2}{a^2} + 3 \cdot \frac{c}{a} \cdot \frac{b}{a} = \frac{b^3}{a^3}$$

i.e. $a^2c + c^2a + 3abc = b^3$. as the required condition.

EXERCISE XV—B

1. Form the equation whose roots are :

(i) 10, 12. (ii) $\frac{1}{3}, \frac{-1}{2}$. (iii) $a+b, a-b$.

(iv) $2 \pm \sqrt{3}$. (v) $2 \pm \sqrt{-3}$.

2. Prove that $1 + \sqrt{2}$ is a root of $x^3 - x^2 - 3x - 1 = 0$ and also find other roots.

3. Form the equation, one of whose roots is

$$\frac{p - \sqrt{p^2 - 4q}}{p + \sqrt{p^2 - 4q}}.$$

[Hint : Rationalise the denominator.]

4. (a) Form the equation whose roots are the squares of the roots of $x^2 - bx - c = 0$.

(b) If α, β are roots of $3x^2 + 2x + 1 = 0$ form an equation whose roots are

$$\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}.$$

5. If α and β are the roots of $px^2 - qx + r = 0$ form the equation whose roots are $\alpha^2 + \beta^2$ and $2\alpha\beta$. (H. S. B. 1951)

6. If α, β are the roots of $2x^2 - 3x - 5 = 0$ form an equation whose roots are

$$\frac{\alpha}{\alpha^2 + \beta^2}, \frac{\beta}{\alpha^2 + \beta^2}. \quad (\text{Q. 1950.})$$

7. Find the equation [whose roots are the cubes of the roots of $x^2 + x + 1 = 0$].

8. If α, β be the roots of $x^2 + px + q = 0$, form the equation whose roots are (i) $\alpha^2 + \alpha\beta$ and $\beta^2 + \alpha\beta$. (ii) $\frac{1}{\alpha + \beta}, \frac{1}{\alpha\beta}$.
(H. S. 1948)

9. If x_1 and x_2 are the roots of the equation $ax^2+bx+c=0$, form the equation whose roots are $(x_1-x_2)^2$ and $(x_1+x_2)^2$.
(P. U. 1946).

10. If the roots of $x^2+bx+c=0$ be α, β and those of $x^2+px+q=0$ be $k\alpha, k\beta$, show that $qb^2=cp^2$.

11. Find the value of m in order that (a) one root of $3x^2-(1+4m)x+m^2+5=0$ may be one-third of the other. (b) the roots of $x^2-2x(1+3m)+7(3+2m)=0$ may be equal.
(H. S. 1947).

12. If the roots of the equation $ax^2+bx+c=0$ be in the ratio of $\gamma : 1$; show that $ac(1+\gamma^2)=\gamma(b^2-2ac)$ and if in the ratio $p : q$ then $ac(p+q)^2=b^2pq$.
(H. S. 1949.)

13. If one root of the equation $ax^2+bx+c=0$ be the square of the other; find the relation between a, b and c . (P.U.)

14. If the equation $px^2+qx+r=0$ is not changed when p, q and r are simultaneously increased by h , show that $x^2+x+1=0$.

15. Find the value of a so that the roots of the equation $3x^2-2x+a=0$ may be reciprocals of each other.

16. If α and β be the roots of $3x^2+x+7=0$ find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
(K.U. 1950).

17. If p, q be the roots of $x^2+bx+c=0$, show that b and c are the roots of $x^2+(p+q-pq)x-pq(p+q)=0$.

18. If α, β are the roots of the equation $ax^2+2bx+c=0$, show that $(n\alpha-\beta)(\alpha-n\beta)=\frac{4nb^2-(n+1)^2ac}{a^2}$. Also show that the roots are real and each lies between half and double of the other provided that $8b^2$ lies between $8ac$ and $9ac$.

[Hint : Take $n=2$ in the above result].

19. Find the condition that the roots of the equation $ax^2+bx+c=0$ shall be (i) reciprocal of each other, (ii) both positive, (iii) both negative (iv) one positive and one negative (v) equal and opposite in sign.

20. Find the conditions that the equation $ax^2+bx+c=0$ may have

(i) One root zero (ii) both roots zero (iii) one root infinite
(iv) both roots infinite (v) one root infinite and other root zero.

[**Hint** : $ax^2+bx+c=0$ has one zero root if $c=0$ and two zero roots when $b=0, c=0$.

If we put $x = \frac{1}{y}$, the equation becomes $cy^2+by+a=0$.

It is obvious that if the y equation has one zero root the x equation has one infinite root and if the y equation has two zero roots, the x equation has two infinite roots. Hence the condition that $ax^2+bx+c=0$ may have two infinite roots is that $a=0, b=0$].

21. Obtain the equation whose roots are $a\alpha-b$ and $\alpha\beta-b$ where α, β are the roots of the quadratic equation $x^2-ax+b=0$. If a and b are such that roots of the given equation are real and positive, prove that the roots of the equation which you have obtained are also real and positive. (D. U. 1943).

22. If α and β be the roots of $ax^2-bx+b=0$ prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{b}{a}} = 0$ and form an equation having $\sqrt{\frac{\alpha}{\beta}}, \sqrt{\frac{\beta}{\alpha}}$ as roots. (K. U. 1951).

3.6. To find the condition that the two equations.
 $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$,
may have a common root.

Let α be the root common to them. Therefore we have

$$a\alpha^2+b\alpha+c=0 \quad \dots(i)$$

$$\text{and} \quad a'\alpha^2+b'\alpha+c'=0 \quad \dots(ii)$$

By cross-multiplication, we get

$$\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{ca'-c'a} = \frac{1}{ab'-a'b}$$

whence we have $(ca'-c'a)^2 = (ab'-a'b)(bc'-b'c)$.
as the required condition.

Cor. If $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ have a common root, then this common root is $\frac{ca'-c'a}{ab'-a'b}$ or which is the same thing, as $\frac{bc'-b'c}{ca'-c'a}$.

3.61. If $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ have both roots common then

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

If α, β be the roots of both these equations ;

$$\text{then } \alpha + \beta = -\frac{b}{a} = -\frac{b'}{a'} \quad \therefore \frac{a}{a'} = \frac{b}{b'}$$

$$\text{Also } \alpha\beta = \frac{c}{a} = \frac{c'}{a'} \quad \therefore \frac{a}{a'} = \frac{c}{c'}$$

$$\text{Hence } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

EXERCISE XVI

1. If $nx^2+mx+1=0$ and $mx^2+nx+1=0$ have a common root, show that either $m=n$ or $m+n+1=0$.

2. If the expressions x^2+9x+m and $x^2+12x+4m$ have a common factor, then show that $m=0$ or 8

3. Find the value of p for which $3x+2$ is a factor of the expression $6x^2+px-6$, and find the other factor when p has that value.

4. If the difference of the corresponding roots of the equations $x^2-px+q=0$ and $x^2-qx+p=0$, is the same, then $p+q+4=0$.

5. If $x^2+ax+bc=0$ and $x^2+bx+ca=0$ have a common root, show that their other roots satisfy the equation $x^2+cx+ab=0$.

6. Show that if $ax^2+bx+c=0$ and $px^2+qx+r=0$ have both the roots common then

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c}.$$

7. If the equations $ax^2+bx+c=0$, $a'x^2+b'x+c'=0$, have one root common, and if the remaining root of the second is the square of the remaining root of the first then show that $a^2c'(ac'-a'c)=a'c^2(a'b-ab')$. (P. U. 1950).

3.7. Complex Numbers. A number of the form $a+ib$ where a and b are two real numbers and $i^2=-1$ is called a complex number (or an imaginary number). Any Complex number of the form $\sqrt{-b^2}$ can be written as ib where b is real. Thus, e. g.

$$\sqrt{-3}=i\sqrt{3}, \sqrt{-12}=i\sqrt{12}=2i\sqrt{3}.$$

The following are some of the *Properties* of complex numbers :—

(i) If $a+ib=0$, then $a=0$ and $b=0$.

For since $a+ib=0$, we get

$$a=-ib \text{ or } a=-b^2$$

$$\text{i. e., } a^2+b^2=0, \therefore a=0, b=0.$$

Cor. 1. If $a+ib=c+id$ then $a=c$ and $b=d$.

Cor. 2. If $a+ib=c+id$ then $a-ib=c-id$.

Cor. 3. The product of two conjugate complex numbers say $c+id$ and $c-id$ is always real, namely, c^2+d^2 .

(ii) To find the product or quotient of two complex numbers in the form $A+iB$.

$$\text{Here } (a+ib)(c+id)=ac+i^2bd+ibc+iad=ac-bd+i(bc+ad)$$

$$\begin{aligned} \text{and } \frac{a+ib}{c+id} &= \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + i \frac{bc-ad}{c^2+d^2}. \end{aligned}$$

3.71. To find all the cube roots of unity and their properties.

$$\text{Let } x=(1)^{\frac{1}{3}}$$

$$\therefore x^3=1 \text{ or } x^3-1=0$$

$$\text{or } (x-1)(x^2+x+1)=0.$$

$$\therefore \text{ either } x=1 \text{ or } x^2+x+1=0$$

$$\text{i. e., } x=1 \text{ and } \frac{-1 \pm \sqrt{1-4}}{2} \text{ or } \frac{-1 \pm \sqrt{-3}}{2}.$$

Thus the three cube roots of unity are

$$1, \frac{-1+i\sqrt{-3}}{2} \text{ and } \frac{-1-i\sqrt{-3}}{2}.$$

Properties. Let the above three cube roots of unity be called $1, \alpha, \beta$; then,

(i) cube of each of these numbers is always unity.

(ii) $\alpha^2 = \beta$ and $\beta^2 = \alpha$

$$\text{for } \alpha^2 = \left(\frac{-1+i\sqrt{3}}{2} \right)^2 = \frac{1-3-2i\sqrt{3}}{4} = \frac{-1-i\sqrt{3}}{2} = \beta.$$

(iii) $\frac{1}{\alpha} = \beta$ or $\frac{1}{\beta} = \alpha$.

$$\begin{aligned} \text{for } \frac{1}{\alpha} &= \frac{2}{-1+i\sqrt{3}} = \frac{2}{-1+i\sqrt{3}} \times \frac{-1-i\sqrt{3}}{-1-i\sqrt{3}} = \frac{2(-1-i\sqrt{3})}{1-(-3)} \\ &= \frac{-1-i\sqrt{3}}{2} = \beta. \end{aligned}$$

(iv) $1 + \alpha + \beta = 0$,

$$\text{for } 1 + \alpha + \beta = 1 + \frac{-1+i\sqrt{3}}{2} + \frac{-1-i\sqrt{3}}{2} = 1 - 1 = 0.$$

3.72. Now if we denote $\frac{-1+i\sqrt{3}}{2}$ by w then by property

(ii) $\frac{-1-i\sqrt{3}}{2}$ is w^2 .

Thus the three cube roots of unity are $1, w, w^2$,

where (i) $w^3 = 1$

and (ii) $1 + w + w^2 = 0$.

Ex. 1. Show that $w^{28} + w^{29} + 1 = 0$.

$$\text{Here } w^{28} = w \times w^{27} = w \times (w^3)^9 = w \times 1 = w$$

$$\text{Similarly } w^{29} = w^2 \times w^{27} = w^2$$

$$\therefore 1 + w^{28} + w^{29} = 1 + w + w^2 = 0.$$

Ex. 2. Prove that $\left(\frac{-1+\sqrt{-3}}{2} \right)^5 + \left(\frac{-1-\sqrt{-3}}{2} \right)^5 = -1$.

$$\text{Here } \left(\frac{-1+\sqrt{-3}}{2} \right)^5 + \left(\frac{-1-\sqrt{-3}}{2} \right)^5$$

$$= w^5 + (w^2)^5 = w^5 + w^{10}$$

$$= w^3 \times w^2 + w^9 \times w = w + w^2 = -1.$$

Ex. 3. Show that $\left[\frac{-1+\sqrt{-3}}{2} \right]^n + \left[\frac{-1-\sqrt{-3}}{2} \right]^n$

is equal to 2, if n be a multiple of 3, and if n be not a multiple of 3 its value is -1 ,

$$\text{Here } \left(\frac{-1 + \sqrt{-3}}{2} \right)^n + \left[\frac{-1 - \sqrt{-3}}{2} \right]^n = w^n + w^{2n}. \quad \dots (i)$$

Now if $n=3p$, where p is an integer (i. e., when n is a multiple of 3).

$$\text{then } w^n + w^{2n} = w^{3p} + w^{6p} = (w^3)^p + (w^3)^{2p} = 1 + 1 = 2.$$

If n is not a multiple of 3, then when divided by 3 it has the remainder 1 or 2, i. e., then

$$\begin{aligned} \therefore \quad w^n + w^{2n} &= w^{3p+1} + w^{6p+2} \quad \text{or } 3p+2 \\ &= w + w^2 \quad \text{or } w^2 + w \\ &= -1. \end{aligned}$$

Ex. 4. Find the cube roots of -1 . (D. U. 1943)

$$\text{Here } x = (-1)^{\frac{1}{3}} \quad \therefore x^3 = -1$$

$$\therefore x^3 + 1 = 0$$

$$\text{or } (x+1)(x^2-x+1)=0$$

$$\therefore \text{ either } x = -1 \quad \text{or } x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$\text{i. e., } x = -1, \text{ or } x = \frac{1 + \sqrt{-3}}{2} \text{ and } x = \frac{1 - \sqrt{-3}}{2}$$

are the required roots.

Ex. 5. Find all the cube roots of 8.

$$\text{Here } x^3 = 8$$

$$\therefore x^3 - 8 = 0$$

$$\text{or } (x-2)(x^2+2x+4)=0$$

$$\therefore x = 2 \quad \text{or } \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm i\sqrt{3}.$$

Evidently the roots are

$$2, 2w, 2w^2.$$

Important : Note that all the cube roots of a number A are

$$A^{\frac{1}{3}}, \quad wA^{\frac{1}{3}}, \quad w^2A^{\frac{1}{3}},$$

where w is an imaginary cube root of unity.

EXERCISE XVII

1. Multiply (i) $\sqrt{-ax}$ by $\sqrt{-ay}$.
 (ii) $(a - \sqrt{-x})$ by $(a + \sqrt{-x})$.
 2. (a) Show that (i) $(3 + 2i)(1 + 3i) = -3 + 11i$.
 (ii) $\frac{2+3i}{4-3i} + \frac{3-i}{4+3i} = \frac{8+5i}{25}$.
 - (b) If $\frac{a+ib}{a-ib}$ be of the form $x+iy$, find x, y . (Q. 1950).
 3. Show that all the roots of $x^4=1$ are $\pm 1, \pm i$.
 4. Show that all the cube roots of
 (i) -8 are $-2, -2\omega, -2\omega^2$. (ii) 27 are $3, 3\omega, 3\omega^2$.
 (iii) 64 are $4, 4\omega, 4\omega^2$, where ω has the usual meaning.
 5. Show that (i) $\left(\frac{-1+\sqrt{-3}}{2}\right)^{98} + \left(\frac{-1-\sqrt{-3}}{2}\right)^{98} = -1$.
 (ii) $[-1+\sqrt{-3}]^{11} + [-1-\sqrt{-3}]^{11} = -2^{11}$.
 6. If $1, \omega, \omega^2$, be the cube roots of unity show that
 (i) $(1-\omega)(1-\omega^2)=3$ (ii) $(1-\omega-\omega^2)^6=64$.
 (iii) $(2-\omega)(2-\omega^2)(2-\omega^3)(2-\omega^4)=49$.
 (iv) $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$.
 7. Show that
 (i) $a^2+ab+b^2=(a-\omega b)(a-\omega^2 b)$.
 (ii) $a^3-b^3=(a-b)(a-\omega b)(a-\omega^2 b)$
 8. Find the value of x^2-2x+7 when $x=1+\sqrt{-2}$.
 9. Obtain the value of $2x^4+5x^3+7x^2-x+37$ when $x=-2-i\sqrt{3}$.
- Hint** :—Since $x=-2-i\sqrt{3}$. $\therefore (x+2)^2=3i^2=-3$
 $\therefore x^2+4x+7=0$ now use it.]

3.8. Nature of the roots. To discuss the nature of the roots of the quadratic equation $ax^2+bx+c=0$.

Solving by the method already given we see that if α, β are roots of this equation, then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \dots(1)$$

and
$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \dots(2)$$

(i) When $b^2 - 4ac$ is positive and not zero; then the roots are real and unequal. Further, if $b^2 - 4ac$ is a perfect square the roots are rational and unequal.

(ii) When $b^2 - 4ac$ is negative, the quantity $\sqrt{b^2 - 4ac}$ is called imaginary, to distinguish it from the square root of positive number which we call real. In this case the roots are imaginary.

(iii) When $b^2 - 4ac$ is equal to zero. Then each of the roots reduces to $-\frac{b}{2a}$ and therefore the roots are equal and real.

(iv) When $b=0$, the equation becomes $ax^2+c=0$, its roots are $\pm \sqrt{-\frac{c}{a}}$ i.e. the roots are equal but of opposite sign when $b=0$.

3.7. It follows from (i) and (ii) that surd roots of an equation with rational co-efficients occur in conjugate pair, i.e., if $c + \sqrt{d}$ be a root then $c - \sqrt{d}$ also is a root, where $\sqrt{d}$ is a proper surd, i.e. where d is not a perfect square.

Imaginary roots of an equation with real co-efficients also occur in conjugate pairs i.e., if $c + \sqrt{-d}$ is a root, then the other root is $c - \sqrt{-d}$ where d is a positive number.

The expression $b^2 - 4ac$ is called the **Discriminant of the equation** $ax^2+bx+c=0$ or of the expression ax^2+bx+c .

Ex. 1. Discuss the nature of the roots of :

(1) $2x^2 - 4x + 5 = 0$.

(2) $3x^2 + 9x + 4 = 0$.

(3) $6x^2 + 13x + 5 = 0$.

(4) $4x^2 - 4x + 1 = 0$.

(1) Here $a=2$, $b=-4$, $c=5$.

$\therefore b^2-4ac=16-40=-24$ which is negative, $\therefore$ the roots are imaginary.

(2) Here b^2-4ac is $81-48=33$, which is positive and not a perfect square. Therefore the roots are real, unequal and irrational.

(3) $a=6$, $b=-13$, $c=5$.

$b^2-4ac=49$ which is positive and a perfect square, $\therefore$ roots are real, rational and unequal.

(3) In this case $b^2-4ac=16-16=0$ $\therefore$ the roots are equal.

Ex. 2. Find for what values of m the equation

$$12x^2+4(m+1)x+3=0 \text{ shall have equal roots.}$$

The roots of the equation will be equal if the discriminant is zero, i.e., if

$$16(m+1)^2-4 \times 12 \times 3=0$$

or if $(m+1)^2-9=0$ or if $(m+1)=\pm 3$ or if $m=2$ or -4 .

Ex. 3. Show that the roots of

$$(b^2-4ac)x^2+4(a+c)x-4=0 \text{ are always real.}$$

The discriminant of the equation

$$=16(a+c)^2+16(b^2-4ac)$$

$$=16[(a+c)^2-4ac+b^2]$$

$$=16[(a-c)^2+b^2] \text{ which is always positive.}$$

Hence the roots are always real.

EXERCISE XVIII—A.

1. Examine the nature of roots of the following equations :—

(i) $3x^2+7x+2=0$.

(ii) $4x^2-9x+3=0$.

(iii) $4x^2-20x+25=0$.

(iv) $(x-a)(x-b)=c^2$.

2. Find the value of a for which the roots of the equation $ax^2+3x-4=0$ are equal.

3. Show that the roots of the equation

$$(a-b)x^2+2(a+b)x-(a-b)=0$$

are real, a, b, c being real, quantities.

Given that a, b, c are real show that the roots of

(i) $(b-c)x^2+(c-a)x+(a-b)=0$

(ii) $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are real.

5. Show that the roots of $2(a^2+b^2)x^2+2(a+b)x+1=0$ are imaginary.
6. Find the values of m for which the equation $4x^2+(1-2m)x+2=0$ has equal roots.
7. For what values of t will the equation $x^2+4x(2+t)+5(3t+11)=0$ has equal roots ?
8. Find k such that $(4-k)x^2+(2k+4)x+(3k+1)=0$ may have equal roots.
9. Find the condition that $m^2x^2+2x(mc-2a)+c^2=0$ may have equal roots
10. Show that $y=mx+\frac{a}{m}$, regarded as an equation in m , will have equal roots if $y^2=4ax$.
11. (i) Prove that the roots of $ax^2+bx+c=0$ are rational, if $a+b+c=0$.
 (ii) Prove that the roots of the equation $(3b-2a)x^2-2(a+3b)x+3b+4a=0$ are rational.
12. (i) Show that the roots of $(x-a)(x-b)=b^2$ are always real. (H. S. 1948)
 (ii) Show that the roots of $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$, are real and that they cannot be equal unless $a=b=c$.
13. If the roots of the equation $ax^2+2bx+c=0$ are real and unequal; then the roots of the equation $x^2+2(a+c)x+(a^2+2b^2+c^2)=0$ are imaginary
14. Prove that the roots of the equation $(a+c-b)x^2+2cx+(b+c-a)=0$ are rational.

3.71. Ex.1. For what values of x is $\frac{x-7}{x^2-2x+4}$ always positive ?

The denominator $x^2-2x+4=(x-1)^2+3$ is positive for all real values of x . Hence the fraction is positive for real values of x if the numerator is positive, i.e., if $x>7$.

Ex. 2. Show that $\frac{x^2+8x+80}{2x+8}$ does not lie between 8 and -8 if x be real.

Let $\frac{x^2+8x+80}{2x+8}=m$, then $x^2+x(8-2m)+80-8m=0$
gives real values of x , if

$$(8-2m)^2 \geq (80-8m) \times 4, \text{ i.e., } (4-m)^2 \geq (10-m)$$

$$\text{i.e., } m^2-64 \geq 0, \text{ i.e., } (m-8)(m+8) \geq 0.$$

This is not true if m lies between -8 and 8 .

Ex. 3. Find the limits between which

$$\frac{1+x+x^2}{1-x+x^2} \text{ must lie when } x \text{ is real.}$$

$$\text{Let } \frac{1+x+x^2}{1-x+x^2}=y.$$

$$\therefore (1+x+x^2)=y(1-x+x^2)$$

$$\text{or } (1-y)x^2+(y+1)x+1-y=0$$

Now this equation in x must have real roots because x is given to be real.

$$\therefore (y+1)^2-4(1-y)^2 \quad \text{is not negative}$$

$$\text{i.e., } -3y^2+10y-3 \quad \text{is not negative}$$

$$\text{or } (3-y)(3y-1) \quad \text{is not negative}$$

$$\text{i.e., } y \text{ must lie between } 3 \text{ and } \frac{1}{3}.$$

EXERCISE XVIII—B

1. If x be real show that
 $3x^2+6x+7$ cannot be less than 4.
2. Show that if x be real then
 $11-6x-3x^2$ cannot be greater than 14.
3. Show that $2x^2-6x+5$ is always positive.
4. Find the values of x for which

$$\frac{x-4}{x^2-3x+5} \text{ is positive}$$

5. Show that $x+\frac{1}{x}$ cannot lie between 2 and -2 for real values of x .
6. Show that $\frac{2x^2-2x-5}{2x-7}$ cannot lie between 11 and 1, if x is restricted to real values only.
7. Prove that $\frac{x^2+4x+2}{11x^2+12x+6}$ lies between $\frac{1}{3}$ and $-\frac{1}{5}$ for real values of x .

8. If x is real, prove that

$$\frac{x^2 - 3x}{3x^2 - 4x + 1} \text{ cannot lie between } \frac{1}{2} \text{ and } \frac{9}{2}.$$

9. Prove that for real values of x ,

$$\frac{x^2 - 2x + 3}{x^2 + 2x + 9} \text{ lies between } \frac{1}{2} \text{ and } 2.$$

10. Show that for real values of x , the expression $\frac{(x-1)(x-2)}{(x-3)(x-4)}$ cannot lie between $-7 + 4\sqrt{3}$ and $-7 - 4\sqrt{3}$.

11. Prove that for real values of x the expression $\frac{3x-4}{x^2+1}$ must lie between $\frac{1}{2}$ and $-\frac{9}{2}$.

3.9. A quadratic expression. We now discuss the nature of the sign of the quadratic expression $ax^2 + bx + c$ in relation to the nature of the roots of the corresponding quadratic equation $ax^2 + bx + c = 0$.

***3.91.** If the roots of the equation $ax^2 + bx + c = 0$ are imaginary or are equal, the expression $ax^2 + bx + c$ for any real value of x , has the same sign as the sign of a or of c .

We have

$$\begin{aligned} ax^2 + bx + c &= a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) \\ &= a \left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} \right] \\ &= a \left[\left(x + \frac{b}{2a} \right)^2 + \left(\frac{4ac - b^2}{4a^2} \right) \right] \quad \dots(1) \end{aligned}$$

Now when the roots of $ax^2 + bx + c = 0$ are imaginary, $b^2 - 4ac$ is negative and therefore $4ac - b^2$ is positive, and consequently $\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2}$ is positive.

Hence the given expression is of the same sign as that of a (i. e. it is positive if a is positive and negative if a is negative).

Also since $4ac - b^2$ is positive therefore ac is positive i. e., a and c have the same sign.

Again when the roots of $ax^2 + bx + c = 0$ are equal $b^2 - 4ac = 0$.

$$\therefore \text{ from (1) } ax^2 + bx + c = a \left[x + \frac{b}{2a} \right]^2.$$

So that $ax^2 + bx + c$ has the same sign as that of a .

Also since $b^2 - 4ac = 0$ or $4ac = b^2$

ac is positive and a and c have the same sign.

Hence in this case also the expression has the same sign as that of a or of c .

3. 92. If α, β are the real roots of the equation $ax^2 + bx + c = 0$, the expression $ax^2 + bx + c$ (i) vanishes when $x = \alpha$ or β and (ii) has the same sign as that of a for all other values of x except those lying between α and β .

(ii) The values α and β satisfy the equation, therefore the expression $ax^2 + bx + c$ is zero when $x = \alpha$ or β .

(ii) We have

$$\begin{aligned} ax^2 + bx + c &= a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) \\ &= a[x^2 - (\alpha + \beta)x + \alpha\beta] \\ &= a(x - \alpha)(x - \beta) \end{aligned} \quad \dots(1)$$

α, β being different, one of them must be greater than the other. Let α be greater than β , then

When $x > \alpha$, $x - \alpha$ is positive, so $x - \beta$ is also positive.

$\therefore$ from (1) $ax^2 + bx + c$ has the same sign as that of a .

When $x < \beta$, $x - \beta$ is negative so $x - \alpha$ is also negative.

$\therefore$ from (1) $ax^2 + bx + c$ has the same sign as that of a .

Lastly, when $x > \beta$ but $< \alpha$, i.e., x lies between α and β then $x - \alpha$ is negative, $x - \beta$ is positive, so that $(x - \alpha)(x - \beta)$ is negative.

$\therefore$ from (1) the sign of $ax^2 + bx + c$ is opposite to that of a (i.e. it is +ve when a is -ve and -ve when a is +ve).

Hence from theorem I and II given above we find that :

The sign of the expression $ax^2 + bx + c$ is opposite to that of a only when the roots of the equation $ax^2 + bx + c = 0$ are real and unequal and x has a value lying between them.

REVISION QUESTIONS I

Solve for x :

$$1. \text{ (i) } \frac{x+2}{x-2} + \frac{2x-3}{2(x-1)} = \frac{23}{6}, \text{ (ii) } \frac{3}{5-x} + \frac{2}{4-x} = \frac{8}{x+2}.$$

2. (i) $(3a^2 + b^2)(x^2 - x - 1) = (3b^2 + a^2)(x^2 + x + 1)$.

(ii) $(a-1)(1+x+x^2)^2 = (a+1)(1+x^2+x^4)$.

3. (i) $\frac{x-a}{x+a} + \frac{x+a}{x-a} = \frac{34}{15}$. (ii) $\sqrt{\frac{x}{x-1}} + \sqrt{\frac{x-1}{x}} = 2\frac{1}{6}$.

4. $\frac{x+2}{x-1} + \frac{2(1+4x)}{1-x^2} + \frac{3}{x+1} = 8$.

5. (i) $\left(x - \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) + 10 = 0$.

(ii) $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$.

6. $x^2 - 3x + \frac{5}{x^2 - 3x} + \frac{9}{2} = 0$.

7. $x^4 + 2ax^3 = 2x + \frac{1}{a^2}$.

[Sol. Given equation is

$$x^4 - \frac{1}{a^2} + 2ax\left(x^2 - \frac{1}{a}\right) = 0$$

or $\left(x^2 - \frac{1}{a}\right)\left(x^2 + \frac{1}{a}\right) + 2ax\left(x^2 - \frac{1}{a}\right) = 0$

$\therefore x^2 - \frac{1}{a} = 0$ i.e. $x = \pm \frac{1}{\sqrt{a}}$, or $x^2 + \frac{1}{a} + 2ax = 0$ which

is easy to solve.]

8. (i) $\left(x - \frac{1}{x}\right)\left(x - \frac{4}{x}\right)\left(x - \frac{9}{x}\right) = (x-1)(x-2)(x-3)$.

(ii) $(b-c)x^2 + (c-a)x + (a-b) = 0$. (P. U. 1922)

9. A square floor is covered with carpet, except a border 18 inches wide round the carpet which is covered with linoleum. The carpet and linoleum cost respectively 5s and 3s. per square yard. The whole cost is £5. 7s. Find the length of the side of the floor.

10. Two bicyclists ride at the rate of 10 m.p.h. The perimeter of the driving wheel of one bicycle is 8 inches less than that of the other, and it turns round once more in 5 seconds. Find the perimeter of driving wheel of each bicycle.

11. A man buys a number of cows for £80. If he had bought 4 more for the same money each cow would have cost £1 less. How many cows did he buy?

12. A man rows to a place $16\frac{1}{2}$ miles down a river and back in 6 hrs. 40 minutes. If the rate of the stream had been, twice as great, he would have taken 6 hrs. $52\frac{1}{2}$ minutes. Find the rates of the stream and the man.

13. Solve :

$$(i) \frac{\sqrt{3x-2}-\sqrt{x+7}}{\sqrt{3x-2}+\sqrt{x+7}} = \frac{x-3}{2x+1}.$$

$$(ii) 9x-3x^2+4\sqrt{x^2-3x+5}=11.$$

$$(iii) \sqrt{x+6}-\sqrt{x-3}=\sqrt{x-18}.$$

$$(iv) \sqrt{ax+b}+\sqrt{ax+c}=\sqrt{2(b+c)}.$$

$$(v) x^2-a(a+1)x+a^3-a^2b+ab-b^2=0.$$

$$(vi) \frac{2(x+a)}{1+5a}=x+5a+\frac{1+3a}{x}.$$

$$(vii) x^2-(a-b)x-1+a-b=0.$$

$$(viii) ab(x^2-1)+x(a^2-b^2)=c[(a-b)x-(a+b-c)].$$

$$(ix) \frac{b}{x+a}+\frac{x+b}{a}+2=0.$$

14. If α, β and γ, δ be the roots of $x^2-px+q=0$ and $x^2-p'x+q'=0$ respectively, find the value of

$$(\alpha-\gamma)(\beta-\delta)+(\beta-\gamma)(\alpha-\delta).$$

15. (i) If α, β be the roots of $x^2-3x+1=0$; find the equation whose roots are $\frac{\alpha}{1+\beta}, \frac{\beta}{1+\alpha}$.

(ii) If α and β be the roots of the equation $ax^2+bx+c=0$, find the equation whose roots are $(\alpha-\beta)^2$ and $(\alpha+\beta)^2$.

16. (a) Solve $x^2+2px=1$ and show that the two roots are real and of opposite signs.

(b) Solve $x+xy=3, y+xy=4$. (B. U.)

17. Find the condition that the roots of $a^2x^2+b^2x+c^2=0$ may be the squares of the roots of $ax^2+bx+c=0$.

18. If one root of $px^2+qx+r=0$ is four times the other, show that $4q^2=25pr$.

19. Prove that the algebraic expression $x^2+2bx+c$ is greater than, equal to or less than $(x+b)^2$ according as the roots of $x^2+2bx+c=0$ are imaginary, equal or real.

20. If the roots of $px^2 + 2qx + q = 0$ are imaginary, show that the roots of $qx^2 + (q-r)x - (r+p-q) = 0$ are real.

21. Find the equation whose roots are the squares of the roots of the equation $px^2 = qx + r$. (P.U.)

22. If x is real, find the greatest and the least values of $\frac{x^2 - 2x + 9}{x^2 + 2x + 9}$ and the greatest value of $\frac{x^2 + 14x + 2}{x^2 + 2x + 3}$.

23. If α, β be the roots of the equation $(x^2 + 1)(a^2 + 1) = abx(ax - 1)$, prove that $(\alpha^2 + 1)(\beta^2 + 1) = b\alpha\beta(\alpha\beta - 1)$.

24. If α, α^2 be the roots of the equation $ax^2 + bx + c = 0$, find c when $a = 8, b = -30$.

25. If α, β be the roots of the equation $x^2 + x + 1 = 0$, form the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ and account for the identity of the equation obtained with the original equation.

26. If α and β be the roots of $x^2 + x + 1 = 0$, show that $\beta^2 = \alpha$.

27. If α, β be the roots of $x^2 + px + q = 0$, find the values of $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)$, where ω is a cube root of unity.

28. Without finding the three cube roots of unity, show that their sum is zero.

Simplify $\left[\frac{-1 + \sqrt{-3}}{2} \right]^8 + \left[\frac{-1 - \sqrt{-3}}{2} \right]^8$.

29. Show that if $b^2 - 4ac$ is positive the expression $ax^2 + bx + c$ cannot keep the same sign for all values of x ; but if $b^2 - 4ac$ is negative then $ax^2 + bx + c$ keeps the same sign as that of a and c .

30. If α, β be the roots of the equation $ax^2 + bx + c = 0$ and α', β' those of $a'x^2 + b'x + c' = 0$, find the condition that

$$\frac{1}{\alpha - \alpha'} + \frac{1}{\alpha - \beta'} = \frac{2}{\alpha - \beta}.$$

31. Find the condition that the two equations $px^2 + qx + r = 0$ and $ax^2 + bx + c = 0$ shall have two roots such that one of them is the reciprocal of the other.

32. Obtain the equation whose roots, are $a\alpha - b$ and $a\beta - b$, where α, β are the roots of the quadratic equation $x^2 - ax + b = 0$.

If a and b are such that the roots of the given equation are real and positive, prove that the roots of the equation that you have obtained are also real and positive.

33. Solve $x^2 + y^2 = a^2$ and $x + y = b$ and show that the sum of the two sides of a right angled triangle cannot exceed $\sqrt{2}$ times the hypotenuse.

34. If one of the roots of $x^2 + ax + b = 0$ is one half of the other, prove that $9b$ equals $2a^2$. (P.U.)

If α, β are the imaginary cube roots of unity, show that $\alpha^n + \beta^n = 2$ or -1 according as n is a multiple of 3 or any other integer.

35. If $x^2 + x = X$ is a quadratic equation in x , find the value of X if one root be $1\frac{1}{2}$. Find also the other root. (P.U.)

36. If α and β be the roots of the equation, $x^2 + 2ax + b = 0$; form the equation whose roots are $\alpha^2 + \alpha\beta + \beta^2, \alpha^4 + \alpha^2\beta^2 + \beta^4$. (P.U. 1927)

37. (a) Obtain the quadratic equation of which the sum of the roots is 1 and the product of the two roots is -1 .

(b) If α and β are the roots of $x^2 + x + 1 = 0$, obtain the equation whose roots are α^3 and β^3 . (B.U.)

38. State and prove the conditions that the roots of the equation $px^2 + qx + r = 0$ may be (i) real and different, (ii) equal and (iii) imaginary.

What values of k will make $(4-k)x^2 + (2k+1)x + 8k+1$, a perfect square?

39. Find the equation whose roots are the squares of the roots of the equation

$$x^2 - x(2\sqrt{c+b})^{\frac{1}{2}} + \sqrt{c} = 0.$$

40. If $\frac{4}{x} = 3 - \sqrt{5}$; show that $x + \frac{4}{x} = 6$ and calculate

the value of $x^2 + \frac{16}{x^2}$.

41. Find the condition that $8x^2 + ax - 3$ and $32x^2 + 2ax - 15$ may have a common factor.

42. If α and β be the roots of $ax^2 + bx + c = 0$,

$$\text{show that } \frac{\beta}{a\alpha + b} + \frac{\alpha}{a\beta + b} + \frac{2}{a} = 0. \quad (\text{C.U.})$$

43. One root of the equation $ax^2 + 2bx + c = 0$ is the reciprocal of one of the roots of the equation $a'x^2 + b'x + c' = 0$.

Show that $(aa' - cc')^2 = 4(ab' - bc')(a'b - b'c)$.

44. If the equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have one root common and if the remaining root of the second is the square of the remaining root of the first then

$$a^2c'(ac' - a'c) = a'c^2(a'b - ab').$$

45. If quadratic equations be formed whose roots differ from those of $x^2 + px + q = 0$ by the same quantity, prove that the discriminants are the same. (D. U. 1948)

CHAPTER IV

SERIES—ARITHMETICAL PROGRESSION

4.1. A set of any number of terms formed and arranged according to some definite law and added together is called a series.

Consider the following :—

(1) 2, 6, 10, 14, 18,.....

(2) $\frac{3}{2}$, 3, 6, 12, 24,.....

(3) $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{11}$,.....

(4) 3, 6, 11, 18, ...

(5) 5, 2, 12, 15, 9,.....

In (1) each term is obtained by adding 4 to the preceding term ; in (2) each term is obtained by multiplying the preceding term by 2 ; in (3) the numerator of each term is 1 and the denominator is obtained by adding 3 to the denominator of the previous term ; in (4) at first sight there appears to be no law of formation but a closer examination shows that the set can be rewritten as

(1^2+2) , (2^2+2) , (3^2+2) , (4^2+2) , ...

and the law of formation is evident ; in (5) there is no law. Hence the first 4 sets form series, while, the fifth one is not a series.

Arithmetical Progression

4.2. A series in which the difference obtained by subtracting any term from the next following term is the same throughout is called an **Arithmetical Progression** ; written briefly as A. P. The constant difference is called the **common difference**.

Consider the following series :—

(1) 1, 4, 7, 10,.....

(2) a , $(a+d)$, $(a+2d)$,.....

(3) 10, 8, 6, 4,.....

In (1) $4-1=7-4=10-7=.....=3$, thus it is an A. P. whose first term is 1 and common difference is 3. In (2) as

before the common difference is d . And in (3) the common difference is -2 .

It may be observed that an A. P. is determined if its first term and the common difference are known; also that any term is obtained from the preceding one by adding the common difference or from the following term by subtracting it. Thus, if l is one term of an A. P. with d as common difference, the preceding term would be $l-d$, and the following term would be $l+d$.

4.3. *Given the first term and the common difference of an A. P. to find the n th term or the **general term**.*

Let a be the first term and d the common difference; then the series is

$$a + (a+d) + (a+2d) + (a+3d) + \dots$$

$$\text{2nd term} = a + d = a + (2-1)d$$

$$\text{3rd term} = a + 2d = a + (3-1)d$$

$$\text{4th term} = a + 3d = a + (4-1)d$$

We notice that the co-efficient of d in any term is always one less than the number of the term in the series. Hence the n th term $= a + (n-1)d$.

Thus the n th term of an A. P. whose first term is a and common difference d is $(a + n - 1d)$.

Ex. 1. Find the 12th term of the series.

$$2, 7, 12, 17, \dots$$

Here $a=2$, $d=5$ and $n=12$.

$$\text{Hence 12th term} = 2 + (12-1)5 = 57.$$

Ex. 2. What term of the series 15, 13, 11, 9, etc., is -83 ?

Let the n th term of the series be -83 .

The first term $= 15$,

the common difference $= 13 - 15 = -2$.

$$\text{Hence the } n\text{th term} = 15 + (n-1)(-2) = 17 - 2n$$

$$\therefore 17 - 2n = -83; \text{ whence } n = 50.$$

Thus -83 is the 50th term of the series.

Ex. 3 If the third term of an A. P. is 7 and the 5th term is 11, find the 20th term.

Let a be the first term and d the common difference.

Then the 3rd term is $a + 2d = 7$.
and the fifth term is $a + 4d = 11$.

Subtracting, we get $2d = 4$ or $d = 2$, and therefore
 $a = 3$.

$\therefore$ 20th term $= a + 19d = 3 + 19 \times 2 = 41$.

Ex. 4. Find the series whose n th term is $2n + 1$.

Put $n = 1$, thus the 1st term $= 2 + 1 = 3$

„ $n = 2$, „ „ 2nd term $= 4 + 1 = 5$

„ $n = 3$, „ „ 3rd term $= 6 + 1 = 7$ and so on.

Hence the series is $3 + 5 + 7 + 9 + \dots$

EXERCISE XX

1. Find (i) the 10th, (ii) 25th, (iii) n th term of the series 115, 112, 109, ...

2. Find the 29th term of the series $-3, -5, -7, -9, \dots$

3. The 6th term of an A. P. is 16 and the 14th term is 32. Find the 29th term.

4. If there be 60 terms in an A. P. of which the first term is 9 and the last term 186, find the 41st term

5. The sum of the 5th and the 12th terms of an A. P. is 81, and the sum of the 8th and the 15th terms is 111; find the n th term.

6. What terms of the series

$9 + 11 + 13 + 15 + \dots$ are 65 and 99?

7. If the 5th and the 14th terms of an A. P. are 3 and -24 respectively, which term of the series is 0?

8. Find the series whose r th term is $3r - 5$.

9. Find the 1st, 2nd, 11th and n th terms of the series whose $(p + 3)$ th term is $5p + 6$.

10. Find how many terms there are in the series $-3 + 3 + 9 + \dots + 117$.

4.4. To find the sum of n terms of an A. P.

Let a be the first term and d the common difference; then the various terms of the A. P. are $a, a + d, a + 2d$, and so on upto n terms.

If l denotes the last term, i. e., the n th term, then

$$l = a + (n - 1)d.$$

.....(1)

The terms of the series in the reverse order are $l, l-d, l-2d$, and so on.

Now if S_n denotes the sum of n terms, we have

$$S_n = a + (a+d) + (a+2d) + \dots + (l-2d) + (l-d) + l.$$

And writing the series in reverse order.

$$S_n = l + (l-d) + (l-2d) + \dots + (a+2d) + (a+d) + a.$$

Adding, we get

$$\begin{aligned} 2S_n &= (a+l) + (a+l) + \dots + (a+l) \\ &= (a+l) \text{ written } n \text{ times} = n(a+l) \end{aligned}$$

$$\therefore S_n = \frac{n}{2} (a+l). \quad \dots\dots I$$

$$= \frac{n}{2} [a + a + (n-1)d] \text{ from (I)}$$

$$= \frac{n}{2} [2a + (n-1)d]. \quad \dots\dots II$$

It may be noted that in formula II four quantities, a, d, n and S_n are involved, and hence when any three of these are given, the fourth can be determined.

Equation I gives the sum of n terms of an A. P. when the first term and the last term are known, while equation II gives the sum when the first term and the common difference are known.

Thus the sum of n terms of an A. P. when first term is a and last term is l is $\frac{n}{2} (a+l)$ and when common difference d is known it is $\frac{n}{2} [2a + (n-1)d]$.

Ex. 1. Find the 17th term and the sum of 17 terms of series $2 + 3\frac{1}{4} + 4\frac{1}{2} + \dots$

$$\begin{aligned} \text{Here } a &= 2, & d &= 3\frac{1}{4} - 2 = 1\frac{1}{4} = \frac{5}{4} \\ & & T_n &= a + (n-1)d. \end{aligned}$$

Putting $n=17$, and substituting for a and d we get

$$T_{17} = 2 + (17-1)\frac{5}{4} = 2 + 16 \times \frac{5}{4} = 2 + 20 = 22$$

$$\text{and } S_n = \frac{n}{2} [2a + (n-1)d] \text{ when } n=17$$

$$\therefore S_{17} = \frac{17}{2} [2 \times 2 + (17-1)\frac{5}{4}]$$

$$= \frac{17}{2} [4 + 20] = 17 \times 12 = 204.$$

The sum may also be found by using formula I

$$\text{i.e., } S = \frac{n}{2} (a + l) = \frac{17}{2} (2 + 22) = 204.$$

Ex. 2. The second term of an A. P. is 15 and the fifth term is double the first. Find the sum of first 20 terms of the progression.

Let the first term be a and the common difference be d .

Then the second term $= a + d = 15$

Fifth term $= a + 4d = 2a$.

These give $a + d = 15$

and $-a + 4d = 0$

$\therefore 5d = 15$

i.e., $d = 3$ and $a = 12$.

Thus sum of first 20 terms

$$= \frac{20}{2} \{2 \times 12 + (20 - 1) \times 3\}$$

$$= 10(24 + 57) = 810.$$

Ex. 3. The first term of an A. P. is -16 and common difference is 4 . Find n , when the sum of n terms is 72 .

Here $a = -16$, $d = 4$ and $S = 72$; substituting these values in formula II, we have

$$72 = \frac{n}{2} [-32 + (n-1)4]$$

$$\text{i.e. } 144 = -32n + 4n^2 - 4n \quad \text{or } n^2 - 9n - 36 = 0$$

$$\text{or } (n-12)(n+3) = 0 \quad \therefore n = 12, \text{ or } n = -3.$$

The first twelve terms of the series are

$-16, -12, -8, -4, 0, 4, 8, 12, 16, 20, 24, 28$, the sum of which is 72 .

If we begin at the last of these terms and count backwards 3 terms, the sum is again 72 ; thus although the negative number $n = -3$ does not answer the question, an intelligent meaning can be assigned to the negative integral value of n .

Ex. 4. Sum the series

$1 + 3 - 5 + 7 + 9 - 11 + 13 + 15 - 17 + \dots$ to $3n$ terms.

The series can be written as

$(1 + 3 - 5) + (7 + 9 - 11) + (13 + 15 - 17) + \dots$ to n terms.

Thus we have to find the result of the following,

$$(1+7+13+\dots\text{to } n \text{ terms}) + (3+9+15+\dots\text{to } n \text{ terms}) \\ - (5+11+17+\dots\text{to } n \text{ terms}).$$

$$= \frac{n}{2} (2 + (n-1)6) + \frac{n}{2} (3 + (n-1)6) - \frac{n}{2} (5 + (n-1)6) \\ = n(3n-2) + n(3n) - n(3n+2), \\ = 3n^2 - 4n.$$

Ex. 5. If the p th term of an A. P. be $3p+2$, find the sum of first ten terms.

Here $T_p = 3p+2$

$\therefore T_1 = 3 \times 1 + 2 = 5$

$T_{10} = 3 \times 10 + 2 = 32$

Hence by I,

$$S_{10} = \frac{10}{2} (5 + 32) = 5 \times 37 = 185.$$

Ex. 6. If the sum of n terms of a series be n^2+n , show that it is an A. P. and find it.

Sum of n terms $= n^2 + n.$

$\therefore$ Sum of $(n-1)$ terms $= (n-1)^2 + (n-1) = n^2 - n$

$\therefore$ n th term $= (n^2 + n) - (n^2 - n) = 2n$

$\therefore$ putting $n=1, 2, 3$ and so on, the series is $2+4+6+8+\dots$ which is an A. P. the common difference being 2.

Remember : n th term = sum of first n term - sum of first $n-1$ terms.

EXERCISE XXI

1. Sum to n terms the series $5+8+11+\dots$
2. The sum of n terms of an A. P. is 40, the common difference 2, and the last term 13; find n .

Sum up :—

3. $\frac{1}{n} + \frac{n+1}{n} + \frac{2n+1}{n} + \dots$ to n terms.

4. $\left(1 - \frac{1}{n}\right) + \left(1 - \frac{2}{n}\right) + \left(1 - \frac{3}{n}\right) + \dots$ to n terms.

5. $(n-1) + (n-2) + (n-3) + \dots$ to n terms.

6. Find how many terms of the A. P. $-3, -1\frac{1}{2}, 0, 1\frac{1}{2}, \dots$ must be taken to give a sum of 225.

7. How many terms must be taken of the series $5+7+9+\dots$ to make the sum 140 ?

Explain the double answer.

8. The sum of n terms of an A. P., whose first term is 5 and common difference 36 is equal to the sum of $2n$ terms of another A. P. whose first term is 36, and common difference 5 ; find n .

9. There are two A. P.'s both commencing with 2, and common difference of the first being twice as great as that of the second, and the sum of the first seven terms of the first being 63 more than that of the second. Find the two series.

10. Find the sum of all numbers less than 500 which are divisible by 5.

11. Find the sum of all the integers from 1 to 200 excluding those that are multiples of 3 or 7.

12. The first term of an A. P. is unity and the ratio of the sum of the first five terms to the sum of the next five terms is 9 : 29. Find the series.

13. Find the sum of n terms of a series whose (i) r th term is $3r-5$ (ii) $(r+3)$ th term is $3r-4$.

14. If the sum of n terms of a series is an^2+bn , find the first term and the common difference.

15. Sum up the series

$$\frac{2x-y}{x+y} + \frac{3x-4y}{x+y} + \frac{4x-7y}{x+y} + \dots \text{ to } (x+y) \text{ terms.}$$

16. Two travellers start together on the same road. One of them travels uniformly 10 miles a day, the other travels 8 miles the first day and increases his pace half a mile a day each succeeding day. After how many days will the latter overtake the former.

17. An athlete during his training increases his daily run by $1\frac{1}{2}$ miles every day ; he starts his training on the first of January and runs $3\frac{1}{2}$ miles on that day. Find (i) the day on which he runs 21 miles (ii) how far he has run from the beginning of his training to the end of that day.

18. Obtain the sum of all numbers in the first 1,000 integers which are neither divisible by 5 nor by 2. (P. U. 1922).

4.5. Arithmetic Mean. If three quantities are in A. P. then the middle one is said to be the **Arithmetic Mean** between the other two.

For the sake of brevity Arithmetic Mean is written A.M.

If the quantities $a, A_1, A_2, \dots, A_n, b$ are in A. P., then $A_1, A_2, \dots, A_n$, are defined as n Arithmetic Means between a and b .

4.6. To find the Arithmetic Mean between two given quantities.

Let the given quantities be a and b and A be the A. M. between them ;

then, since a, A, b are in A. P.

$\therefore$ common difference $= A - a = b - A$.

i.e., $2A = a + b$.

or $A = \frac{a+b}{2}$.

Thus the Arithmetic Mean of two given quantities is half the sum of the two quantities.

Note 1.—To show that x, y, z are in A. P., it is enough to show that $2y = x + z$.

Definition. The Arithmetic Mean of n quantities $x_1, x_2, x_3, \dots, x_n$ is defined to be

$$\frac{x_1 + x_2 + x_3 + \dots + x_n}{n}$$

4.7. To find n Arithmetic Means between two given quantities.

Let the given quantities be a and b ; and let the n Arithmetic Means be

$$A_1, A_2, \dots, A_n.$$

Hence, $a, A_1, A_2, \dots, A_n, b$ are in A.P.

Then a is the first term and b is the $(n+2)$ th term.

$\therefore b = a + (n+2-1)d = a + (n+1)d$,

or

$$d = \frac{b-a}{n+1} :$$

$$\therefore A_1 = a + d = a + \frac{b-a}{n+1}$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1} \text{ and so on.}$$

$$A_n = a + nd = a + \frac{n(b-a)}{n+1}$$

Ex. 1. Insert 3 Arithmetic Means between 3 and $\frac{177}{11}$.

Let the means be A_1, A_2 , and A_3 .

Then 3, $A_1, A_2, A_3, \frac{177}{11}$ are in A. P.

Let the common difference be d so that the first term is 3 and 5th. term is $\frac{177}{11}$.

$$\therefore \frac{177}{11} = 3 + (5-1)d = 3 + 4d. \text{ i. e., } 4d = \frac{177}{11} - 3$$

$$\therefore d = \frac{177 - 33}{11 \times 4} = \frac{36}{44} = \frac{9}{11}$$

$$\therefore A_1 = 3 + d = 3 + \frac{9}{11} = \frac{42}{11};$$

$$\therefore A_2 = 3 + 2d = 3 + \frac{18}{11} = \frac{51}{11};$$

$$A_3 = 3 + 3d = 3 + \frac{27}{11} = \frac{60}{11}.$$

Ex. 2. Find the 14 arithmetic means which can be inserted between 5 and 8 and show that their sum is 14 times the arithmetic mean between 5 and 8.

Let d be the common difference of the Arithmetic Progression beginning with 5 and ending with eight when the 14 means have been inserted.

Then the last term 8 is the 16th term of the series.

$$\therefore 8 = 5 + (16-1)d, \text{ i. e., } d = \frac{3}{15} = \frac{1}{5}.$$

Thus the means are

$$5 + \frac{1}{5}, 5 + \frac{2}{5}, 5 + \frac{3}{5}, \dots, 5 + \frac{14}{5}.$$

$$\text{The sum of 14 means} = \frac{14}{2} \left\{ \left(5 + \frac{1}{5} \right) + \left(5 + \frac{14}{5} \right) \right\}$$

$$= \frac{14}{2} \left\{ 13 \right\} = 14 \left(\frac{13}{2} \right).$$

$$= \text{but the Arithmetic mean between 5 and 8} = \frac{5+8}{2} = \frac{13}{2}$$

Hence the sum of the 14 means inserted above = 14 times the arithmetic mean between 5 and 8.

Ex. 3. Insert n Arithmetic Means between n and $2n$.

Let d be the common difference, then

$$2n = (n+2)\text{th term} = n + (n+2-1)d$$

$$\therefore d = \frac{n}{n+1}$$

Thus n A. M's. between n and $2n$ can be written down.

EXERCISE XXII

1. Insert 8 Arithmetic means between $-3\frac{1}{2}$ and 4.
2. Insert five Arithmetic Means between 20 and 17.
3. Insert p A. M.s. between p^2 and 1.
4. Insert 8 Arithmetic Means between $5x-4y$ and $5y-4x$.
5. Show that the sum of n Arithmetic Means between a and b is $\frac{n(a+b)}{2}$ i.e. n times the A. M. between a and b .

6. Find the ratio of the sum of n Arithmetic means between a and b to the sum of m Arithmetic Means between the same.

4.8. Miscellaneous Methods and examples.

Important Note :— It may be observed that whenever we have to take an odd number of terms in A. P. it is convenient to take the middle term as a and common difference as d ; while if we have to take an even number of terms in A. P., then it is convenient to take the two middle terms as $a-d$ and $a+d$ and $2d$ as the common difference. For example six terms would be taken as $a-5d, a-3d, a-d, a+d, a+3d, a+5d$.

Ex. 1. Three numbers in A. P are such that their sum is 18 and their product is 192. Find the numbers.

Let the three numbers be $a-d, a, a+d$.

Then from given conditions we have

$$(a-d) + a + (a+d) = 18, \text{ i.e., } 3a = 18 \text{ or } a = 6$$

and $a(a-d)(a+d) = 192$, or $a^2 - d^2 = 192/6 = 32$

or $d^2 = 36 - 32 = 4$, $\therefore d = \pm 2$.

The numbers therefore are 4, 6, 8 or 8, 6, 4.

Ex. 2. The sum of four numbers in A. P. is 16 and sum of their squares is 84. Find the numbers.

Let the four numbers in A. P. be $a-3d, a-d, a+d, a+3d$. Then their sum being $4a$, we get

$$4a = 16, \text{ i.e., } a = 4.$$

Also $(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 84$

or $4a^2 + 20d^2 = 84$,

$$20d^2 = 84 - 4 \times 16 = 20,$$

$$d = \pm 1.$$

$\therefore$ the numbers are 1, 3, 5, 7.

Ex. 3. The p th term of an A. P. is a and q th term is b . Show that the sum of the $(p+q)$ terms is

$$\frac{p+q}{2} \left[a+b + \frac{a-b}{p-q} \right].$$

Let the first term be a and common difference be β .

Then $T_p = a + (p-1)\beta = a$ (i)

$$T_q = a + (q-1)\beta = b$$
 (ii)

Subtracting we get

$$b-a = (q-p)\beta$$

$$\therefore \beta = \frac{b-a}{q-p} = \frac{a-b}{p-q}$$
 (iii)

Adding (i) and (ii) we get

$$a+b = 2a + (p+q-2)\beta$$
 (iv)

Now $S_{p+q} = \frac{p+q}{2} [2a + (p+q-1)\beta]$

$$= \frac{p+q}{2} [2a + (p+q-2)\beta + \beta]$$

$$= \frac{p+q}{2} \left[a+b + \frac{a-b}{p-q} \right] \text{ [from (iii) and (iv)]}.$$

Ex. 4. The sum of r terms of an A. P. is $4r^2 + 3r$. Find the p th term.

Here $S_r = 4r^2 + 3r$, so $S_{r-1} = 4(r-1)^2 + 3(r-1)$.

$$\therefore T_r = S_r - S_{r-1} = 4[r^2 - (r-1)^2] + 3 = 4(2r-1) + 3 = 8r-1.$$

$$\therefore T_p = 8p-1.$$

Ex. 5. If a^2 , b^2 and c^2 are in A. P., show that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.} \quad (\text{H. S. 1949})$$

Now $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.

$$\text{If } \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a},$$

$$\text{or if } \frac{(b-a)}{(c+a)(b+c)} = \frac{(c-b)}{(c+a)(a+b)},$$

$$\text{or if } b^2 - a^2 = c^2 - b^2$$

$$\text{or if } a^2, b^2, \text{ and } c^2 \text{ are in A. P.}$$

Hence the result follows.

Ex. 6. The sum of n terms of each of the two A. P.'s are in the ratio $3n+4 : 5n+6$, find the ratio of their 11th terms.

Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two A. P.'s.

$\therefore$ the ratio of 11th terms

$$= \frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{2a_1 + 20d_1}{2a_2 + 20d_2} = \frac{2a_1 + (21-1)d_1}{2a_2 + (21-1)d_2}$$

$$= \frac{21}{2} [2a_1 + (21-1)d_1]$$

$$= \frac{21}{2} [2a_2 + (21-1)d_2]$$

Sum of first 21 terms of one A. P.

$$= \frac{\text{Sum of first 21 terms of second A. P.}}{\text{Sum of first 21 terms of second A. P.}}$$

$$= \frac{3 \times 21 + 4}{5 \times 21 + 6} \left(\because \frac{S_n}{S'_n} = \frac{3n+4}{5n+6} \text{ and here } n=21. \right)$$

$$= \frac{67}{111}$$

Ex. 7. If a, b , and c are in A.P., prove that
 $a^2(b+c) + b^2(c+a) + c^2(a+b) = \frac{2}{9}(a+b+c)^3$.

Since a, b, c are in A.P., let $a = l - m, b = l, c = l + m$

$$\begin{aligned}\therefore \text{L.H.S.} &= (l-m)^2(2l+m) + l^2 \times 2l + (l+m)^2(2l-m) \\ &= 2l[(l+m)^2 + (l-m)^2] - m[(l+m)^2 - (l-m)^2] + 2l^3 \\ &= 4l^3 + 4lm^2 - 4lm^2 + 2l^3 = 6l^3.\end{aligned}$$

$$\text{R.H.S.} = \frac{2}{9}(l-m+l+l+m)^3 = \frac{2}{9} \times 27l^3 = 6l^3.$$

Ex. 8. Find n such that

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} \text{ may be A. M. between } a \text{ and } b.$$

Here A. M. is $\frac{a+b}{2}$.

$$\therefore \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{a+b}{2}$$

$$\text{or } 2a^{n+1} + 2b^{n+1} = a^{n+1} + ab^n + ba^n + b^{n+1}$$

$$\text{or } a^n(a-b) = b^n(a-b)$$

$$\therefore a^n = b^n \text{ since } a \neq b$$

$$\text{i. e., } \left(\frac{a}{b}\right)^n = 1$$

$$\therefore n = 0.$$

EXERCISE XXIII

1. The last term of an A. P. is ten times the first, and the last but one is equal to the sum of 4th and 5th. Find the number of terms and show that the common difference is equal to the first term.

2. The sum of three numbers in A. P. is 24 and their product is 440; find the numbers.

3. The sum of three terms of an A. P. is 15 and the sum of their squares is 83. Find the numbers.

4. The sum of three numbers in A. P. is 27 and the sum of the squares of the two extremes is 170; find the numbers.

5. Four numbers are in A. P. Their sum is 24 and their product is 945. Find the numbers.

6. Four numbers in A. P. are such that the sum of the two extremes is 14 and the product of the means is 45 ; find the numbers.

7. Divide 32 into four parts which are in A. P. such that the product of the first and fourth is to the product of the second and third in the ratio of 7 to 15.

8. The sides of a right-angled triangle are in A. P., show that they are proportional to the numbers 3, 4, 5.

9. Show that if unity be added to the sum of any number of terms of the series $8 + 16 + 24 + \dots$ the result is the square of an odd number.

10. If the sum of n terms of a series be a quadratic expression in n , prove that the series is an A. P. and find the first four terms of a series the sum of whose n terms is $4n^2 + 3$.

11. The sum of a series upto n terms is $2n^2 + n + 5$. Find the series. Is it an A. P.

12. Find the first five terms of a series of which the sum to n terms is $n^2 + 2n$,

13. If a, b and c the x th, y th and z th terms of an A. P., show that $a(y-z) + b(z-x) + c(x-y) = 0$.

14. If $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A. P., show that a, b and c are in A. P.

15. If a, b and c are in A. P., show that

$$\frac{ab+ac}{bc}, \frac{bc+ab}{ca}, \frac{ca+cb}{ab} \text{ are in A.P.}$$

16. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P., show that

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in A.P.}$$

17. If a, b and c are in A. P., prove that

$$a^2(b+c), b^2(c+a), c^2(a+b) \text{ are also in A. P.}$$

Also that if $a(b+c), b(c+a), c(a+b)$ are in A. P.,

$$\text{then } \frac{1}{a}, \frac{1}{b} \text{ and } \frac{1}{c} \text{ are in A. P.}$$

18. If the p th term of an A. P. is q and q th term is p , find the m th term.

19. If n times the n th term of an A. P. is equal to m times the m th term, prove that $(m+n)$ th term is zero.

20. The sum of p terms of an A. P. is q , and the sum of q terms is p , show that the sum of $p+q$ term is $-(p+q)$.

21. If the sum of the first p terms of an A. P. is zero; show that the sum of the next q terms is $-\frac{aq(p+q)}{p-1}$, a being the first term.

22. If S_1, S_2, S_3 be sums of $n, 2n$ and $3n$ terms respectively of an arithmetical progression, show that

$$S_3 = 3(S_2 - S_1).$$

23. Prove that if the number of terms of an A. P. be odd, the sum of the first term and the last term is twice the middle term, and the sum of the series is equal to middle term multiplied by the number of terms.

[Hint Let the number of terms be $2n+1$, so that middle term is $(n+1)$ th term.]

24. If a be the first term and l the last term of an A. P. prove that the sum of the 10th terms from the beginning and the 10th term from the end is $a+l$. Show also that the sum of two terms equidistant from the beginning and the end is constant.

25. An A. P. contains $2n+1$ terms. Prove that n times the sum of the first, third, fifth, etc., terms is equal to $(n+1)$ times the sum of the second, fourth, etc., terms.

26. There are two series in A. P. the sums of which to n terms are as $13-7n : 3n+1$. Prove that their first terms are as $3 : 2$ and their second terms are as $-4 : 5$.

27. The first, second and last terms of an A. P. are respectively a, b, c , show that its sum is $\frac{(a+c)(b+c-2a)}{2(b-a)}$.

CHAPTER V

GEOMETRICAL PROGRESSION

5.1. A series in which the ratio of any term to the preceding term is constant throughout is called a **Geometrical Progression**. This constant ratio is called the **common ratio**.

For brevity, a geometrical progression is written as G. P.

Consider the series :—

$$(1) 1 + 3 + 9 + 27 + \dots \dots$$

$$(2) 2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \dots \dots$$

$$(3) a + ar + ar^2 + \dots \dots \dots$$

In the first series any term divided by the preceding term gives 3 and therefore the series is a G. P. whose common ratio is 3. In the second series the common ratio is $-\frac{1}{4}$ and in the third the common ratio is r .

It is clear that any term is obtained from the preceding term by multiplying it with the common ratio, or from the following term by dividing it.

Thus, if one term of a G. P. is l , and common ratio is r , the following term would be lr and preceding term would be $\frac{l}{r}$.

5.2. Given the first term and the common ratio of a G. P. to find the general term or the n th term.

Let a be the first term and r the common ratio of G. P. Then

$$\text{2nd term} = ar = ar^{2-1}$$

$$\text{3rd term} = ar^2 = ar^{3-1}$$

$$\text{4th term} = ar^3 = ar^{4-1}$$

We notice that in any term the index of r is one less than the number of the term in the series.

$$\text{Hence the } n\text{th term} = ar^{n-1}.$$

Thus the n th term of a G. P., whose first term is a and common ratio is r is ar^{n-1} .

Observe that a G. P. is given if its first term and the common ratio are known.

Ex. 1 Find the 6th term of a G. P. whose first term is 72 and common ratio is $\frac{1}{3}$.

Here $a=72$, $r=\frac{1}{3}$ and $n=6$.

$$\therefore \text{6th term} = ar^{n-1} = 72 \times \left(\frac{1}{3}\right)^5 = \frac{72}{3^5} = \frac{8}{9}.$$

Ex. 2. Find the series in G. P. whose 5th term is 80 and 8th term is 640.

Let a be the first term, and r the common ratio ; then

$$T_5 = ar^4 = 80, \text{ and } T_8 = ar^7 = 640.$$

$$\therefore \text{by division } r^3 = \frac{640}{80} = 8, \text{ i.e., } r = 2;$$

and since $ar^4 = 80$, we get $16a = 80$ or $a = 5$.

Thus the series is $5 + 10 + 20 + 40 + \dots$

Ex. 3. Develop the series when its r th term is ap^{2r-1} .

Evidently the series is obtained by putting $r=1, 2, 3, \dots$

Hence the various terms are $ap, ap^3, ap^5, \dots$

EXERCISE XXIV

- Find the 6th term of $5, \frac{10}{3}, \frac{20}{9}, \dots$
- Find the 7th and the n th terms of the series $3\frac{3}{8}, 2\frac{1}{4}, 1\frac{1}{2}, \dots$
- Find the 10th term of the series in G. P., whose 5th term is 48 and 8th term is -384 .
- Find the 6th term of a series whose n th term is 3^n .
- What term of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ is $\frac{1}{512}$?
- What term of the series $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots$ is $-\frac{1}{256}$?
- The 3rd term of a G. P. is $6\frac{1}{4}$ and the 7th term is the reciprocal of the 3rd term. Which term of the series is 1?
- Find the value of missing terms of the series.
 $? \quad 6 \quad ? \quad ? \quad ? \quad 486.$
 (i) If it is an A. P. ; (ii) if it is a G. P.
- The third term of a G. P. is the square of the first, and the 5th term is 64 ; find the series. (H.S.B. 1946)

10. Prove that a series formed by the reciprocal of each term in G. P. is also a G. P. and its common ratio is the reciprocal of the common ratio of original series.

5.3. To find the sum of n terms of a G. P.

Let a be the first term and r the common ratio, and S_n the sum upto n terms ; then

$$S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$$

$$\therefore rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$$

Subtracting, we get

$$S_n(1-r) = a - ar^n$$

$$\text{i.e., } S_n = \frac{a(1-r^n)}{1-r} \text{ or } \frac{a(r^n-1)}{r-1}, \text{ provided } r \neq 1.$$

Thus the sum to n terms of a G. P. whose first term is a and common ratio is r , is $\frac{a(1-r^n)}{1-r}$.

Note. When $r=1$, $S_n = a + a + a + \dots +$ to n terms, so that $S_n = na$.

Cor. Since the last term $l = ar^{n-1}$, we have $lr = ar^n$

$$S_n = \frac{a - rl}{1-r} = \frac{rl - a}{r-1}.$$

Ex. The second term of a G. P. is $\frac{2}{3}$ and the fifth term is $\frac{16}{81}$. Find the 7th term and sum up to 7 terms.

Let a be the first term and r the common ratio.

$$\text{Then } ar = \frac{2}{3} \text{ and } ar^4 = \frac{16}{81}.$$

$$\text{Dividing we get } r^3 = \frac{8}{27}, \text{ or } r = \frac{2}{3} \text{ and } \therefore a = 1.$$

$$\therefore \text{ the 7th term } = 1 \cdot \left(\frac{2}{3}\right)^6 = \frac{64}{729}.$$

$$\text{Sum up to 7 terms} = \frac{1 \times \left[1 - \left(\frac{2}{3}\right)^7\right]}{1 - \frac{2}{3}}.$$

$$= \frac{1-128}{\frac{1}{3}} = 3 - \frac{128}{729} = 2\frac{601}{729}.$$

EXERCISE XXV

Sum the series :

1. $1\frac{1}{2} + 2\frac{1}{4} + 3\frac{3}{8} + \dots$ up to 8 terms.

2. $\frac{1}{\sqrt{2}} - 1 + \sqrt{2} - \dots$ up to 10 terms.

3. $0.9 + .09 + .009 + \dots$ to n terms.

4. If the sum of n terms of a G. P. is 728, the common ratio being 3 and the first term being 2, find n .

5. In a G. P. the first term is 3, the last but one term is 192 and the last term is -384. Find the sum of the series.

6. Find the sum of the series :

$$(x+a) + (x^2+2a) + (x^3+3a) + \dots \text{ to } n \text{ terms.}$$

7. A rubber ball thrown vertically into the air 60 feet, falls and rebounds 40 feet. It falls again and rebounds $26\frac{2}{3}$ feet, and so on. What distance has the ball traversed up to the instant when it hits the ground for the tenth time ?

8. The sum of $\sqrt{3} + 3 + 3\sqrt{3} + \dots$ is $39 + 13\sqrt{3}$, find the number of terms.
(K. U. 1951)

5.4. Infinite Series : So far we have been dealing with a G. P. consisting of a finite number of terms. We shall now consider series in G. P. where number of terms is *not finite* ; such series are called *infinite series*.

For example, let us consider the un-ending series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ in which the common ratio is $\frac{1}{2}$, which is less than unity, and therefore the terms go on decreasing successively.

Sum to n terms, or S_n is given by

$$S_n = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}, \text{ so that } S_n \text{ is less than } 2 \text{ by } \frac{1}{2^{n-1}}.$$

As n becomes larger and larger $\frac{1}{2^{n-1}}$ becomes smaller and smaller, and therefore the difference between S_n and 2 becomes smaller and smaller, so that when n is very large the difference between S_n and 2 is very small. This is expressed by saying that the limiting sum of the above infinite series is 2.

But if we start summing the terms from the beginning, it is clear that the process of adding up will never stop, as the number of terms is unlimited ; and therefore we cannot find the sum of the series in the ordinary sense of the word 'sum'.

As an illustration, let us assume that in the above series we are dealing with rupees. Then adding up ;

	Rs.	as.	p.
$S_1 = \text{Rs. } 1$	= 1	0	0
$S_2 = \text{Rs. } (1 + \frac{1}{2})$	= 1	8	0
$S_3 = \text{Rs. } (1 + \frac{1}{2} + \frac{1}{4})$	= 1	12	0
$S_4 = \text{Rs. } (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8})$	= 1	14	0
$S_5 = \text{Rs. } (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16})$	= 1	15	0
$S_6 = \text{Rs. } (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32})$	= 1	15	6
.....			
		↓	
		Rs. 2.	

Also we find that

S_1	< Rs 2	by	Rs. 1
S_2	"	"	8 annas
S_3	"	"	4 "
S_4	"	"	2 "
S_5	"	"	1 anna
S_6	"	"	6 pies
.....			
			↓
			0

From this it follows that as we continue adding up, the difference between the sum obtained and Rs. 2 continues diminishing ; also each term that we add fills up only half of the deficiency ; so that however large the number of terms may be taken, the deficiency will never be filled up i.e., the sum will never be Rs. 2. But as the process of adding continues, the deficiency constantly decreases and can be made as small as we like. This phenomenon is described by saying that the limiting sum of the above infinite series is Rs. 2.

In the following article we take up the general G. P. continued to infinity.

5.5. To find the limiting sum of an infinite series in G. P. whose common ratio is numerically less than one.

Let a be the first term and r the common ratio, $r < 1$ numerically.

The sum S_n up to n terms, we know, is given by

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now, when r is numerically less than one, r^n decreases as n increases; so that when n becomes larger and larger $\frac{ar^n}{1-r}$ becomes smaller and smaller and consequently the

difference between S_n and $\frac{a}{1-r}$ can be made as small as we like. The limiting sum of the above series, therefore, is $\frac{a}{1-r}$.

Thus the sum to infinity of a G. P. whose first term is a and common difference is r (which is numerically < 1) is $\frac{a}{1-r}$.

Note 1.—When the limiting sum of an infinite series is known to be finite, the series is said to be convergent. Thus, the infinite series in G. P. is convergent if the common ratio is less than one numerically.

Note 2.—The limiting sum of an infinite series is, for the sake of brevity called its sum and the process of finding it is called summing it up.

Ex. 1. Sum up to infinity $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots$

The sum is $\frac{a}{1-r}$, where $a = \frac{2}{3}$ and $r = \frac{2}{3}$:

$$\therefore \text{Sum} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2.$$

Ex. 2. Find the value of $\cdot 2\dot{3}\dot{7}$.

$\cdot 2\dot{3}\dot{7}$ means $\cdot 2373737\dots$ to infinity.

$$\therefore .2\dot{3}\dot{7} = .2 + .037 + .00037 + .0000037 + \dots$$

$$= \frac{2}{10} + \frac{37}{1000} + \frac{37}{100000} + \dots \text{to infinity,}$$

which excluding the first term, is a G. P. whose common ratio is $\frac{1}{100}$.

$$\begin{aligned} \therefore .2\dot{3}\dot{7} &= \frac{2}{10} + \frac{\frac{37}{1000}}{1 - \frac{1}{100}} = \frac{2}{10} + \frac{37}{990} = \frac{198 + 37}{990} \\ &= \frac{235}{990}. \end{aligned}$$

Observe that the same result is obtained by the usual mechanical formula for recurring decimals in arithmetic: in fact the mechanical formula in arithmetic is based on this principle. In the next example we proceed to prove this.

Ex. 3. Find the value of $\cdot NRRR\dots$ to infinity where N contains n digits and R contains r digits.

$$\text{Let } S = \cdot NRRR\dots \text{to infinity.} \quad (i)$$

$$10^{n+r}S = NR \cdot RRR\dots \text{to infinity.}$$

$$10^n S = N \cdot RRR\dots \text{to infinity.} \quad \begin{matrix} [(i) \times 10^{n+r}] \\ [(i) \times 10^n] \end{matrix}$$

$$\text{Subtracting, } (10^{n+r} - 10^n)S = NR - N.$$

$$\text{Hence } S = \frac{NR - N}{10^n(10^r - 1)}.$$

But $10^r - 1$ is a number consisting of r nines and 10^n represents a unit followed by n ciphers.

EXERCISE XXVI

Sum up the following series to infinity:

$$1. \quad 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$$

$$2. \quad 2 - 1 + \frac{1}{2} - \frac{1}{4} + \dots \quad 3. \quad 3\sqrt{2} + 3 + \frac{3}{\sqrt{2}} + \dots$$

$$4. \quad a + 1 + \frac{1}{a} + \frac{1}{a^2} + \dots \text{(where } a \text{ is numerically } > 1.)$$

$$5. \quad 1 - 2x + 4x^2 - 8x^3 + \dots \text{(} x > \frac{1}{2} \text{ numerically.)}$$

6. $\frac{2}{3} + \frac{5}{9} + \frac{2}{9} + \frac{5}{27} + \frac{2}{27} + \dots$

7. The first term of an infinite geometric series is 2 and the sum to infinity is 3. Find the common ratio and the sum to n terms.

8. Find the common ratio of a G. P. whose first term is 5 and whose sum to infinity is 15.

9. If $y = 1 + x + x^2 + \dots$ ($x < 1$ numerically), show that

$$x = \frac{y-1}{y}$$

10. Simplify :

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{8}} \dots \text{to infinity.}$$

11. Find the value of the following recurring decimals by summing up a G. P. :

(i) $\cdot 5$ (ii) $10\cdot 0\dot{1}$ (iii) $\cdot 23\dot{4}1$.

5.6. If three quantities a , G and b , are in G. P., then G is defined to be the **Geometric Mean** between a and b .

If the quantities a , $g_1, g_2, \dots, g_n, b$ are in G. P. then $g_1, g_2, \dots$ and g_n are said to be n **Geometric Means** between a and b .

For the sake of brevity Geometric Mean is written as G. M.

5.7. To find the G. M. between two given quantities.

Let the given quantities be a and b and G be the Geometric Mean between them.

Then a , G and b are in G. P.

$$\therefore \frac{G}{a} = \frac{b}{G}, \text{ i.e. } G^2 = ab,$$

or $G = \sqrt{ab}.$

Thus the **Geometric Mean** of two quantities is the square root of their product.

Note. To show that x, y, z are in G. P., it is enough to show that $y^2 = xz$.

Definition. The geometric mean of n quantities

$$x_1, x_2, x_3, \dots, x_n \text{ is } (x_1 x_2 x_3 \dots x_n)^{\frac{1}{n}}.$$

58. To insert n Geometric Means between two given quantities.

Let the given quantities be a and b and let the required Geometric Means be $g_1, g_2, \dots, g_n$.

Then by definition $a, g_1, g_2, \dots, g_n, b$ are in G. P.

Let the common ratio be r .

Then the $(n+2)$ th term of this G. P. is b .

$$\therefore ar^{n+2-1} = b, \text{ i.e., } ar^{n+1} = b, \text{ or } r^{n+1} = \frac{b}{a}.$$

$$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \therefore g_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$g_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \dots \dots g_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

5.81. Product of n Geometric Means between a and b .

(P. U. 1947.)

From Art. 58, we have

$$\begin{aligned} g_1 \times g_2 \times \dots \times g_n &= a^n \left(\frac{b}{a}\right)^{\frac{1+2+\dots+n}{n+1}} = a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}} \\ &= a^n \left(\frac{b}{a}\right)^{\frac{n}{2}} = (ab)^{\frac{n}{2}} \\ &= (\sqrt{ab})^n, \end{aligned}$$

Thus the product of n Geometric Means between a and b is equal to the n th power of the Geometric Means between a and b .

Ex. Find the G. M. between $\frac{2}{3}$ and 486 and also insert 5 Geometric Means between them.

Let G be the G. M. between $\frac{2}{3}$ and 486

Then $\frac{2}{3}, G$, and 486 are in G. P.

$$\therefore G = \sqrt{\frac{2}{3} \times 486} = \sqrt{324} = 18,$$

Let g_1, g_2, g_3, g_4 and g_5 be the five Geometric Means between $\frac{2}{3}$ and 486.

Then, by definition $\frac{2}{3}, g_1, g_2, g_3, g_4, g_5, 486$ are in G. P.

Let r be the common ratio.

Since 486 is the 7th term of this series

$$\therefore \frac{2}{3}r^{7-1} = 486 \quad \text{i.e., } r^6 = 243 \times 3 = 3^6, \text{ or } r = 3.$$

$$\therefore g_1 = \frac{2}{3}r = \frac{2}{3} \times 3 = 2$$

$$g_2 = \frac{2}{3}r^2 = \frac{2}{3} \times 9 = 6$$

$$g_3 = \frac{2}{3}r^3 = \frac{2}{3} \times 27 = 18$$

$$g_4 = \frac{2}{3}r^4 = \frac{2}{3} \times 81 = 54$$

$$\text{and } g_5 = \frac{2}{3}r^5 = \frac{2}{3} \times 243 = 162.$$

EXERCISE XXVII ✓

1. Insert 3 geometric means between 3 and 48.

2. Insert 4 geometric means between $\frac{2}{3}$ and $\frac{81}{16}$.

3. Insert 6 geometric means between 28 and $-\frac{7}{32}$.

4. Find n such that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b .

5. The geometric mean of two numbers is 8 and the arithmetic mean is 17. Find the numbers.

6. If the first and last of five geometric means between two numbers be $-\frac{1}{2}$, $-\frac{1}{32}$; find the numbers.

7. Insert between 4 and 18 two numbers, such that the first three may be in A. P. and the last three in G. P.

5.9. Miscellaneous Examples :

Important Note :—When we have to take an odd number of terms in G. P. it will be convenient to take as the common ratio. Similarly when we have to take an even number of terms in G. P. it will be convenient to take the two middle terms as $\frac{a}{r}$ and ar and r^2 as the common ratio.

Thus 3 terms in G. P. may be taken as $\frac{a}{r}$, a and ar ; while 4 terms in G. P. may be taken as

$$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3.$$

Ex. 1 Find the numbers in G. P. whose product is 27 and whose sum is 13.

Let the three numbers be $\frac{a}{r}$, a and ar .

Then $\frac{a}{r} \times a \times ar = 27$, i. e., $a^3 = 27$, or $a = 3$.

$$\begin{aligned}\text{Now } x^{q-r} &= a^{q-r} l^{(p-1)(q-r)} \\ y^{r-p} &= a^{r-p} l^{(q-1)(r-p)} \\ z^{p-q} &= a^{p-q} l^{(r-1)(p-q)}\end{aligned}$$

$$\begin{aligned}\therefore x^{q-r} y^{r-p} z^{p-q} &= a^{q-r+p-q+p-q} \times l^{\sum (p-1)(p-r)} \\ &= a^0 l^0 = 1.\end{aligned}$$

Ex. 7. From three numbers in G. P. three other numbers in G. P. are subtracted and the remainders are still in G. P. Prove that the three series must have the same common ratio.

Let the first G. P. be a, ar, ar^2 and the second be b, bm, bm^2

$\therefore$ the series formed by their differences is —

$$a-b, ar-bm, ar^2-bm^2$$

Now these are in G. P. if $(ar-bm)^2 = (a-b)(ar^2-bm^2)$

i. e. if $ab(r-m)^2 = 0$.

or if $r=m$. So that first two series have the same ratio.

Now for the third,

$$\text{the ratio} = \frac{ar-bm}{a-b} = \frac{(a-b)r}{a-b} = r.$$

Thus the three series have the same ratio.

EXERCISE XXVIII

1. The product of 3 numbers in G. P. is 64 and the sum of their products in pairs is 56. Find the numbers.

2. The product of three numbers in G. P. is 216 and the sum of their products in pairs is 126, find the numbers.

3. Find three numbers in G. P. whose sum is 19 and the sum of whose squares is 133. (P. U. 1946.)

4. A number consists of three digits in geometric sequence. The sum of the digits is 14 and if 594 be added to the number, the digits will be reversed. Find the number.

5. Three numbers whose sum is 15 are in A. P.; if 1, 4 and 19 be added to them respectively, the numbers are in G. P. Find the numbers.

6. If $a, b, c, d, \dots$ be $(n+1)$ quantities in G. P. show that $a^2-b^2, b^2-c^2, c^2-d^2, \dots$ are also in G. P., and find the sum of the n terms of this series.

7. In an A. P. the common difference exceeds the first term by 12; the first term increased by 1, the second term increased by 2, and the third term increased by 1 are in G. P. Find the two possible values of the common difference.

8. The sum of the first five terms of a G. P. is 9 and the sum of the terms from the 5th to the 9th, both inclusive is 729; find the two possible values for the common ratio

9. (i) Find the sum of $(a+b)^2 - (a^2 - b^2) + (a-b)^2 - \dots$ to infinity, where a and b are both positive or both negative.

(ii) Find an infinite G. P. whose sum is 49 and the sum of whose first two terms is 45. Show that there are two such progressions, and write down the first 4 terms of each.

Sum to n terms the series ;

10. $4 + 44 + 444 + \dots$

11. $.3 + .33 + .333 + \dots$

12. Sum to n terms the series whose n th term is $3^n + 2^n$.

13. If a, b, c and d are in G. P., prove that $a+b, b+c, c+d$ are in G. P.

14. If a, b, c and d are in G. P., show that $(a+b)^2, (b+c)^2, (c+d)^2$ are in G. P. and so are their reciprocals.

15. If a, b, c and d are in G. P., show that $(a+b+c+d)^2 = (a+b)^2 + 2(b+c)^2 + (c+d)^2$.

16. (i) If a, b, c be in A. P. and a, b, d be in G. P., show that $a, a-b, d-c$ are in G. P.

(ii) If a, b, c be in G. P. and x, y be A. M.s between a, b and b, c respectively, prove that

$$\frac{a}{x} + \frac{b}{y} = 2; \frac{1}{x} + \frac{1}{y} = \frac{2}{b}$$

(K. U. 1950)

17. The first term of a convergent series in G. P. is a and the sum to infinity is b . Find the sum upto n terms.

18. The first term of a geometric series continued to infinity is 1 and every term is equal to the sum of all the succeeding terms. Find the series.

19. The sum of an infinite G. P. is 8, and the sum of the squares of its terms is 36. Find the series.

20. An A. P. and a G. P. begin with the same term; the sum of their first terms is 4, of their second terms is 10 and of their third terms is 24. Find the common difference and the common ratio.

21. The first term of an A. P. and a G. P. are each equal to 3, and the common difference of the former is twice the common ratio of the latter. the third term of the G. P. is greater than the third term of the A. P. by 1. Find the two possible values of the common difference and the common ratio.

22. The sum of an infinite G. P. is S and the sum of the squares of its terms is s . Show that the first term and the common ratio, are respectively,

$$\frac{2Ss}{S^2+s} \text{ and } \frac{S^2-s}{S^2+s}.$$

23. If x, y, z be respectively the p th, q th and r th terms of a G. P., show that $x^{q-r}y^{r-p}z^{p-q}=1$.

24. If P be the product of n terms in a G. P., S their sum and R the sum of their reciprocals, show that $P^2R^n=S^n$.

25. If p is the product of n terms in a G. P. of which a is the first term and l the last term, prove that $p^2=a^n/n$.

26. In a geometrical progression, show that the product of any two terms equidistant from a given term is equal to the square of the given term.

27. If the p th term of a G. P. is x and q th term is y , find the r th term.

28. Sum up the series

$$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\dots \text{to } n \text{ terms.}$$

29. A marble ball falls from a height of 16 ft. and after striking the floor it rebounds to three-fourths of the height it fell from and so on. Find the limit of total distance travelled by the ball before coming to rest.

30. A lamp hanging at the end of a chain is pulled to the right, one foot out of the vertical and is then released. It swings to a point 0.9 ft. to the left, then .81 to the right of the vertical and so on. Find the limit of the total distance the lamp describes before coming to rest.

31. Show that the amounts of a certain sum of money for various years at compound interest form a G. P. Find, to the nearest rupee, the amount of Rs. 500 in 4 years at the rate of 10 per cent.

32. If $a_1, a_2, \dots, a_n$ be a G. P., show that $\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}$ is also a G. P. If S be the sum of first n terms of first G. P. and S' be the sum of first n terms of second G. P. then show that $\frac{S}{S'}=a_1a_n$.

33. Find the sum of the infinite series

$$1+(1+b)r+(1+b+b^2)r^2+(1+b+b^2+b^3)r^3+\dots$$

r and b being proper fractions.

(P. U. 1950)

CHAPTER VI

HARMONIC PROGRESSION

6.1. A series is said to be in **Harmonic Progression** when the reciprocals of the terms form an arithmetical progression.

For example the series

$$(1) \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$(2) \quad \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \dots$$

$$(3) \quad \frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \dots$$

are obviously in harmonic progression, for the series formed by the reciprocal of the terms of these series, *viz.*

$$(i) \quad 1 + 2 + 3 + 4 + \dots$$

$$(ii) \quad \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \frac{9}{2} + \dots$$

$$(iii) \quad a + (a+d) + (a+2d) + \dots \text{ are in A. P.}$$

It follows, therefore, that if a series be in arithmetical progression, the reciprocal of the terms will form a harmonic progression.

Note.—For brevity, harmonic progression is written as H. P.

6.2. If a, b, c , be in H.P. show that $\frac{a-b}{b-c} = \frac{a}{c}$.

Since a, b, c are in H.P. $\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

$$\therefore \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}, \text{ or } \frac{a-b}{ab} = \frac{b-c}{bc}$$

$$\text{Hence } \frac{a-b}{b-c} = \frac{ab}{bc} = \frac{a}{c}.$$

6.21. Working the same process backwards it can be shown that if $\frac{a-b}{b-c} = \frac{a}{c}$ then a, b, c are in H. P. Consequently a harmonic progression may also be defined as a series in which every three consecutive terms satisfy this relation namely the

ratio of the first and the third terms is equal to the ratio of the difference of the first and the second to the difference of second and third terms.

Ex. Find the first term of an H. P. whose second term is $\frac{3}{5}$ and the third term $\frac{1}{2}$.

Let a be the first term, then $a, \frac{3}{5}, \frac{1}{2}$ are in H. P.

$$\therefore \frac{a}{\frac{3}{5}} = \frac{a - \frac{3}{5}}{\frac{3}{5} - \frac{1}{2}} = \frac{a - \frac{3}{5}}{\frac{1}{10}}$$

$$\text{i.e., } 2a = 10(a - \frac{3}{5}) = 10a - 6 \text{ or } a = \frac{3}{4}.$$

6.3. To Find n th term of an H. P.

From the definition of H. P. it is evident that n th term of an H. P. is the reciprocal of the n th term of the A. P. formed by taking the reciprocals of the terms of H. P.

Thus if $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \dots$ be an H. P.,

then its n th term is $\frac{1}{a + (n-1)d}$.

Ex. 1. Find the 8th term of the series

$$\frac{2}{13}, \frac{1}{6}, \frac{2}{11}, \dots$$

Here reciprocals of the terms are $\frac{13}{2}, \frac{6}{1}, \frac{11}{2}, \dots$ this is an A. P. whose common difference is $-\frac{1}{2}$.

$$\therefore \text{8th term of this A. P.} = \frac{13}{2} + (8-1)(-\frac{1}{2}) = 3$$

$$\therefore \text{8th term of the given series} = \frac{1}{3}.$$

Ex. 2. Find the n th term of the series :

$$\frac{x+y}{2x-y} + \frac{x+y}{3x-4y} + \frac{x+y}{4x-7y} + \dots$$

Here the corresponding A. P. is

$$\frac{2x-y}{x+y} + \frac{3x-4y}{x+y} + \frac{4x-7y}{x+y} + \dots$$

Whose common difference is $\frac{x-3y}{x+y}$

$$\therefore \text{its } n\text{th term} = \frac{2x-y + (n-1)(x-3y)}{x+y}$$

$\therefore$ the required n th term of the given H. P.

$$= \frac{x+y}{2x-y+(n-1)(x-3y)}$$

$$= \frac{x+y}{(n+1)x-(3n-2)y}$$

6.31. Sum of any number of terms of an H. P:—

The student should carefully note that the sum of first n terms of an H. P. is not equal to the reciprocal of sum of first n terms of the corresponding A. P. There is no formula to find the sum of n terms of an H. P. In numerical examples we can only sum up the terms actually.

Example :— Find the sum of first six terms of the series :

$$\frac{2}{5}, \frac{2}{3}, 2, -2, \dots$$

Here corresponding A. P. is

$$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \dots \text{ whose common difference is } -1.$$

$\therefore$ first six terms of A. P. are

$$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}$$

$\therefore$ Sum of the first six terms of the given H. P.

$$= \frac{2}{5} + \frac{2}{3} + 2 - 2 - \frac{2}{3} - \frac{2}{5}$$

$$= 0$$

Notice that this sum is not the sum of the reciprocal of the corresponding six terms in A. P.

6.4. If a, c, b are three quantities in H. P. then c is said to be the **Harmonic Mean** between a and b .

If $a, H_1, H_2, \dots, H_n, b$ are in H. P. then $H_1, H_2, \dots, H_n$ are defined to be n Harmonic Means between a and b .

6.5. To find the Harmonic Mean between two given numbers.

Let a, b be the given numbers and let H be the Harmonic Mean between them.

Then since a, H and b are in H. P.

$$\frac{1}{a}, \frac{1}{H}, \frac{1}{b} \text{ are in A. P.}$$

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

or
$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$$

$$H = \frac{2ab}{a+b}$$

Thus the H. M. between two quantities a and b is $\frac{2ab}{a+b}$.

6.6. To insert n Harmonic Means between two given numbers a and b .

This is easily done by finding the n arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$ and then taking their reciprocals.

Example. Find (i) the harmonic mean between 2 and 3, (ii) and also insert 3 harmonic means between 5 and 6.

(i) Let H be the harmonic mean between 2 and 3. Then 2, H , 3 are in H. P. and therefore,

$$\frac{1}{2}, \frac{1}{H}, \frac{1}{3} \text{ are in A. P.}$$

$$\therefore \frac{1}{H} - \frac{1}{2} = \frac{1}{3} - \frac{1}{H} \text{ or } \frac{2}{H} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$\therefore H = \frac{12}{5}.$$

Otherwise thus : Harmonic mean between a and b is

$$\frac{2ab}{a+b}. \text{ Here } a=2, \text{ and } b=3.$$

$$\therefore \text{Harmonic mean is } \frac{2 \times 2 \times 3}{2+3} = \frac{12}{5}.$$

(ii) Let us insert 3 arithmetic means between $\frac{1}{5}$ and $\frac{1}{6}$ and take their reciprocals.

Let A_1, A_2, A_3 , be the arithmetic means between $\frac{1}{5}$ and $\frac{1}{6}$.

$\therefore \frac{1}{5}, A_1, A_2, A_3, \frac{1}{6}$ are in A. P. Let the common difference be d . Then since $\frac{1}{6}$, is the 5th term of this A. P.,

$$\therefore \frac{1}{5} + (5-1)d = \frac{1}{6} \text{ or } 4d = \frac{1}{6} - \frac{1}{5} = \frac{-1}{30}$$

$$\therefore d = \frac{-1}{120}.$$

$$\therefore A_1 = \frac{1}{5} + d = \frac{1}{5} - \frac{1}{120} = \frac{23}{120}.$$

$$A_2 = \frac{1}{5} + 2d = \frac{1}{5} - \frac{1}{60} = \frac{11}{60}.$$

$$A_3 = \frac{1}{5} + 3d = \frac{1}{5} - \frac{1}{40} = \frac{7}{40}.$$

$\therefore$ the required harmonic means are

$$\frac{120}{23}, \frac{60}{11}, \frac{40}{7}.$$

EXERCISE XXIX

1. The second term of an H. P. is $\frac{1}{3}$ and the 5th term is $\frac{1}{5}$. Find the sixth term of the series.

2. The third and the seventh terms of an harmonic series are $\frac{7}{11}$ and $\frac{7}{31}$. Find the first term and the eighth term.

3. Extend the series $\frac{8}{7} + 2 + 8$ one term each way.

4. Find the n th term of the series

$$3 + \frac{3}{4} + \frac{3}{7} + \dots$$

5. Find the n th term of the series $4 + 4\frac{2}{7} + 4\frac{8}{13} + 5 \dots$

6. Find the harmonic mean between 3 and -5.

7. Insert four harmonic means between $\frac{1}{3}$ and $\frac{1}{13}$.

8. Find x such that $2+x$, $3+x$ and $5+x$ may be in H. P.

9. The harmonic mean between two numbers is 3 and the arithmetic mean is 4. Find the numbers.

10. Find two numbers whose geometric mean is 6 and whose harmonic mean is $\frac{24}{5}$.

11. If the harmonic mean between two numbers is to their geometric mean as 20 to 29, prove that the numbers are in the ratio of 4 to 25.

12. Insert between $\frac{1}{8}$ and $\frac{1}{16}$ two numbers such that the first three may be in H. P. and the last three in G. P.

13. If H be the harmonic mean between a and b , show that

$$\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}.$$

6.7. If A , G and H are respectively the Arithmetic, the Geometric and the Harmonic Means between any two positive numbers, to show that

(i) A , G and H are in G. P. and (ii) $A > G > H$.

Let the two positive numbers be a and b .

$$\text{Then, } A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b};$$

$$\therefore A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2.$$

$\therefore A, G, H$ are in G. P.

$$\begin{aligned} \text{Also } A - G &= \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} \\ &= \frac{(\sqrt{a}-\sqrt{b})^2}{2}. \end{aligned}$$

Which is positive.

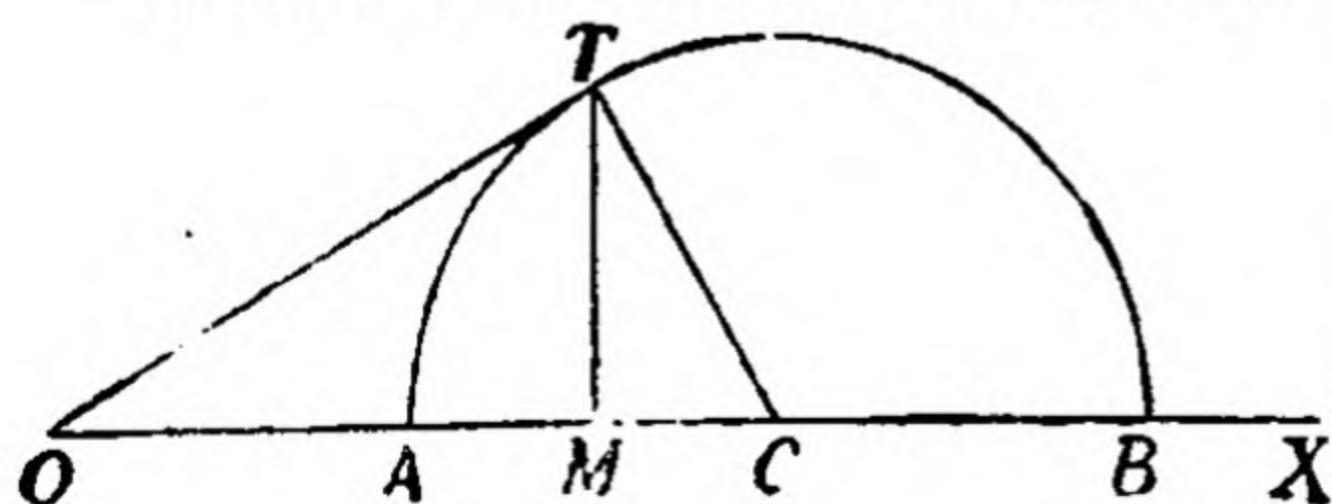
$$\therefore A > G.$$

$$\text{Also since } \frac{G}{A} = \frac{H}{G}, \text{ and } A > G, \therefore G > H.$$

$$\therefore A > G > H.$$

6.7.1. Geometrical representation of A the arithmetic, G the geometric and H , the harmonic means between any two positive numbers a and b .

From a line OX cut off OA and OB , a and b units respectively. Draw



a semi-circle on AB as diameter and draw OT tangent to the circle and TM perp. to AB . Let C be the centre of the circle.

Then $OA = a$, $OB = b$.

$$\therefore OC = \frac{1}{2}(OA + OB) = \frac{a+b}{2} = A. \quad \dots\dots(1)$$

$$OT^2 = OA \times OB = ab = G^2. \quad \dots\dots(2)$$

Also, since triangles OCT and OMT are similar

$$\frac{OM}{OT} = \frac{OT}{OC},$$

$$\text{or } OM = \frac{OT^2}{OC} = \frac{ab}{\frac{a+b}{2}} = \frac{2ab}{a+b} = H. \quad \dots\dots(3)$$

Thus OC, OT and OM are respectively the Arithmetic, Geometric and Harmonic means between the two numbers represented by OA and OB.

• Also it is evident from the figure that

$$OC > OT > OM.$$

$$\text{i.e., } A > G > H.$$

6.8. Miscellaneous Examples.

Ex. 1. $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in H. P., show that $b+c, c+a, a+b$ are in H. P.

Since $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in H. P., therefore a^2, b^2, c^2 are in A. P.

Now $b+c, c+a, a+b$ are in H. P.

if $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.

$$\text{i.e., if } \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\text{or if } \frac{b-a}{b+c} = \frac{c-b}{a+b} \text{ or if } b^2 - a^2 = c^2 - b^2$$

i.e., if a^2, b^2, c^2 are in A. P., which is given.

Ex. 2. The sum of the reciprocals of three numbers in H. P. is 15 and the product of the numbers is $\frac{1}{105}$. Find the numbers.

Let the three numbers in H. P. be

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

$$\therefore (a-d) + a + (a+d) = 15, \quad \text{i.e., } 3a = 15 \quad \therefore a = 5;$$

$$\text{also } \frac{1}{a-d} \times \frac{1}{a} \times \frac{1}{a+d} = \frac{1}{105} \quad \text{i.e., } \frac{1}{5(25-d^2)} = \frac{1}{105}$$

$$\therefore 25 - d^2 = 21 \text{ or } d^2 = 4 \quad \therefore d = \pm 2,$$

Hence the numbers are $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$.

Ex. 3. If a, b, c, d be four numbers in H. P., then show that $a+d > b+c$. (P. U.)

Here because a, b, c, d are in H. P.,

$$\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \text{ are in A. P.}$$

$$\text{i.e., if } \frac{1}{a} = \alpha$$

$$\text{then } \frac{1}{b} = \alpha + \beta, \beta \text{ being the common difference,}$$

$$\frac{1}{c} = \alpha + 2\beta \text{ and } \frac{1}{d} = \alpha + 3\beta.$$

$$\therefore a+d = \frac{1}{\alpha} + \frac{1}{\alpha+3\beta} = \frac{2\alpha+3\beta}{\alpha(\alpha+3\beta)} = \frac{2\alpha+3\beta}{\alpha^2+3\alpha\beta}$$

$$b+c = \frac{1}{\alpha+\beta} + \frac{1}{\alpha+2\beta} = \frac{2\alpha+3\beta}{(\alpha+\beta)(\alpha+2\beta)} = \frac{2\alpha+3\beta}{\alpha^2+3\alpha\beta+2\beta^2}$$

The numerators being the same and the denominator of $b+c$ being greater than that of $a+d$ by $2\beta^2$, a positive quantity, we have

$$a+d > b+c.$$

Ex. 4. Find n such that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the H. M. between a and b .

$$\text{Here } \frac{a^{n+1}+b^{n+1}}{a^n+b^n} \text{ is H. M. between } a \text{ and } b.$$

$$\text{if } \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b},$$

$$\text{or if } (a^{n+1}+b^{n+1})(a+b) = 2ab(a^n+b^n)$$

$$\text{or if } a^{n+2} - ab^{n+1} + b^{n+2} - ba^{n+1} = 0, \text{ i.e., if } (a-b)(a^{n+1} - b^{n+1}) = 0$$

$$\text{Now } \because b \neq a \therefore a^{n+1} = b^{n+1} \text{ or } \left(\frac{a}{b}\right)^{n+1} = 1$$

$$\therefore n+1=0, \text{ i.e., } n=-1.$$

EXERCISE XXX

1. The sum of the reciprocals of three numbers in H.P. is 12 and the product of the numbers is $\frac{1}{48}$. Find the numbers.

2. The sum of three numbers in H. P. is $\frac{37}{80}$ and the first number is $\frac{1}{4}$. Find the numbers.

3. Show that the three numbers a, b, c are in A. P., G. P., or H. P., according as

$$\frac{a-b}{b-c} = \frac{a}{a} \text{ or } = \frac{a}{b}, \text{ or } = \frac{a}{c} \text{ respectively.}$$

4. If a, b and c be in A. P. and x, y , and z be in H. P., show that

$$\frac{a+c}{by} = \frac{x+z}{zx}.$$

5. If a, b, c are in H. P., show that

$$\frac{1}{a} + \frac{1}{b+c}, \frac{1}{b} + \frac{1}{c+a}, \frac{1}{c} + \frac{1}{a+b} \text{ are also in H. P.}$$

6. Find two numbers whose A. M. is greater by 13 than their G. M. and whose G. M. is greater by 12 than their H. M.

7. If $\frac{a+b}{2}, b, \frac{b+c}{2}$ be in H. P. then prove that a, b, c are in G. P.

8. If a, b, c are three positive quantities, and x, y, z the geometric means between a and b, b and c and c and a respectively; if $a, a+x, a+y$ are in G. P. then show that $x, y, -z$ are in H. P.

9. Find the n th term of an H. P. whose first term is a , whose last term is c , and whose number of terms is m .

10. If p th, q th and r th terms of an H. P. be a, b and c respectively, prove that

$$(q-r)bc + (r-p)ca + (p-q)ab = 0.$$

11. If $a-p, b-p, c-p$ be in G. P., show that $2(b-p)$ is the H. M. between $b-a$ and $b-c$.

12. If x_1, x_2 be two arithmetic means, y_1, y_2 the two harmonic means and z_1, z_2 the two geometric means between any two numbers, show that

$$x_1 y_2 = x_2 y_1 = z_1 z_2.$$

13. If p, q, r be in A. P., prove that

$$\frac{qr}{r^2 + pq}, \frac{rp}{pq + qr}, \frac{pq}{qr + rp} \text{ are in H. P.}$$

14. If $b+c, c+a, a+b$ are in H. P. then prove that a^2, b^2, c^2 are in A. P. (H. S. B 1949)

15. (i) If a^2, b^2, c^2 are in A. P. prove that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.}$$

(P. U.)

(ii) If a, b, c , be in H. P. prove that $a(b+c), b(c+a), c(a+b)$ are in A. P. (H. S. B. 1951)

16. If a, b, c be in A. P. and a^2, b^2, c^2 be in H. P. prove that $-\frac{a}{2}, b, c$ are in G. P. or else $a=b=c$.

17. If x, y, z be in A. P. ax, by, cz in G. P. and a, b, c in H. P., prove that

$$\frac{x}{z} + \frac{z}{x} = \frac{a}{c} + \frac{c}{a}.$$

18. If a, b, c be in A. P., b, c, d , in G. P. and c, d, e , in H. P., prove that a, c, e , are in G. P. (Q. 1951) (H. S. B. 1947)

19. If a, b, c are in A. P., b, c, a are in G. P., show that c, a, b are in H. P.

20. If $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ and p, q, r be in A. P., show that, x, y, z are in H. P.

21. If a, b, c be in H. P., show that $2a-b, b$ and $2c-b$ are in G. P.

22. If the p th term of an H. P. be q and the q th term be p , show that the $(p+q)$ th term is $\frac{pq}{p+q}$.

23. (i) If a, b and c be in H. P., show that

$$\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}.$$

(ii) If $a^p = b^q = c^r$, and a, b, c are in G. P., show that p, q, r are in H. P. (Q. 1950).

24. If a, b, c be in H. P. show that if $a > c$,

$$\frac{1}{b-c} + \frac{1}{a-b} > \frac{4}{a-c}.$$

25. Between two different numbers two H. M's. are inserted and one of them is equal to the geometric mean of the numbers. Prove that one of the numbers is equal to four times the other.

26. If between any two numbers there be inserted two arithmetic means A_1, A_2 , two geometric means G_1, G_2 and two harmonic means H_1, H_2 , show that

$$\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2}.$$

CHAPTER VII
MISCELLANEOUS SERIES

We shall now consider some examples of *Summation of Series*, which are neither arithmetical, nor geometrical, nor harmonical. In most of these series the method is to find the *general term*, which indicates the law of formation of the series.

7.1. Arithmetico-Geometric Series. A series each term of which is formed by multiplying the corresponding terms of an A. P. and a G. P. is called an Arithmetico-Geometric series; for example the series,

$a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \dots \{a + (n - 1)d\}r^{n-1}$,
each term of which is formed of the product of the each corresponding terms of the two series.

and $a + (a + d) + (a + 2d) + \dots + (a + n - 1d)$
 $1 + r + r^2 + \dots + r^{n-1}.$

Note.—If in the Arithmetico-Geometric series we put $r=1$ we get A. P. and if we put $d=0$ we get a G. P.

7.2. To find the sum of an Arithmetico-Geometric Series up to n terms.

Let S_n be the sum of the series

$$\begin{aligned} \text{Let } S_n \text{ be the sum of the series} \\ a + (a+d)r + (a+2d)r^2 + \dots + (a+n-1d)r^{n-1} \\ S_n = a + (a+d)r + (a+2d)r^2 + \dots + (a+n-1d)r^{n-1} \\ r \times S_n = ar + (a+d)r^2 + \dots + (a+n-2d)r^{n-1} + (a+n-1d)r^n \end{aligned}$$

$$\therefore S_n(1-r) = a + dr + dr^2 + \dots + dr^{n-1} - (a + n-1)d)r^n$$

$$= a + \frac{d(1-r^n)}{1-r} - (a + n-1)d)r^n$$

$$\therefore S_n = \frac{a}{1-r} + \frac{d_r(1-r^{n-1})}{(1-r)^2} - \frac{(a+n-1d)r^n}{1-r}$$

Cor. If $r < 1$ numerically, the sum of the above series continued to infinity is $\frac{a}{1-r} + \frac{dr}{(1-r)^2}$, for, $r^{n-1}, r^n \rightarrow 0$.

Ex. 1. Find the sum of the series $1+4x+7x^2+\dots$ to n terms; hence write down the sum to infinity when $x < 1$ numerically.

$$\begin{aligned} \text{Let } S_n &= 1 + 4x + 7x^2 + \dots + (3n-5)x^{n-2} + (3n-2)x^{n-1} \\ xS_n &= x + 4x^2 + \dots + (3n-5)x^{n-1} + (3n-2)x^n \\ \therefore S_n(1-x) &= 1 + 3x + 3x^2 + \dots + 3x^{n-1} - (3n-2)x^n \end{aligned}$$

$$= 1 + \frac{3x(1-x^{n-1})}{(1-x)} - (3n-2)x^n$$

$$\therefore S_n = \frac{1}{1-x} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-2)x^n}{1-x}$$

Since $x < 1$ and $n \rightarrow \infty$, $x^{n-1}, x^n \rightarrow 0$

$$\therefore \text{Sum to infinity} = \frac{1}{1-x} + \frac{3x}{(1-x)^2}$$

Ex. 2. Sum up the series

$1+3x+5x^2+7x^3+\dots$ to infinity, $x < 1$ numerically.

$$\text{Let } S = 1 + 3x + 5x^2 + 7x^3 + \dots$$

$$\therefore xS = x + 3x^2 + 5x^3 + \dots$$

$$\therefore S(1-x) = 1 + 2x + 2x^2 + 2x^3 + \dots$$

$$= 1 + \frac{2x}{1-x}$$

$$\therefore S = \frac{1}{1-x} + \frac{2x}{(1-x)^2} = \frac{1-x+2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}$$

Ex. 3 Sum to infinity

$1^2+2^2x+3^2x^2+4^2x^3+\dots$ when $x < 1$ numerically.

Though this is not an Arithmetico-Geometric series but it can be summed by a somewhat similar method.

Let S be the required sum,

$$\text{then } S = 1 + 4x + 9x^2 + 16x^3 + \dots$$

$$\therefore xS = x + 4x^2 + 9x^3 + \dots$$

$$\therefore S(1-x) = 1 + 3x + 5x^2 + 7x^3 + \dots$$

Multiplying by x again, we have

$$x(1-x)S = x + 3x^2 + 5x^3 + \dots$$

$\therefore$ by subtraction

$$S(1-x)^2 = 1 + 2x + 2x^2 + 2x^3 + \dots$$

$$= 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$$

$$\therefore S = \frac{1+x}{(1-x)^3}$$

EXERCISE XXXI

Sum up to n terms the series :

1. $1 + 2x + 3x^2 + 4x^3 + \dots$

2. $1 + 3x + 5x^2 + \dots$

3. $2 \cdot 1 + 3 \cdot 2 + 4 \cdot 4 + 5 \cdot 8 + \dots$

(H.S.B. 1947)

Sum up to infinity the series :

4. $\frac{1}{2} + \frac{3}{6} + \frac{5}{18} + \dots$

5. $\frac{3}{2} - \frac{5}{6} + \frac{7}{18} - \dots$

6. $1 - \frac{2}{5} + \frac{3}{5^2} - \frac{4}{5^3} + \dots$

7. Sum up the series, $\frac{2}{3} + \frac{5}{9} + \frac{8}{27} + \frac{11}{81} + \dots$ to n terms and find the sum to infinity.

Sum up to infinity, given that $x < 1$ numerically.

8. (i) $1 + 4x^2 + 7x^4 + \dots$

(ii) $1 + 3x + 6x^2 + 10x^3 + \dots$

(Q. 1951)

9. $1 - x + 2x^2 - 3x^3 + 4x^4 - \dots$

10. $1^2 + 3^2x + 5^2x^2 + 7^2x^3 + \dots$

11. Find the sum of infinite terms of the series

$$1 + \frac{x}{a}(1+x) + \frac{x^2}{a^2}(1+x+x^2) + \frac{x^3}{a^3}(1+x+x^2+x^3) + \dots$$

12. Show that $2^{\frac{1}{2}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times \dots = 2$.

13. Find the square root of :

$$3^{\frac{1}{2}} \times 9^{\frac{1}{4}} \times 27^{\frac{1}{8}} \times 81^{\frac{1}{16}} \dots \text{ to infinity.}$$

7.3. To find the sum of the first n natural numbers.

Let $S = 1 + 2 + 3 + \dots + n$.

It is an A.P. whose first term is 1 and common difference is 1.

$$\therefore S = \frac{n[2 \times 1 + (n-1)]}{2} = \frac{n(n+1)}{2}$$

Thus the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Note: —Using the sigma notation, this may be written as

$$\Sigma n = \frac{n(n+1)}{2}$$

7.4. To find the sum of the squares of the first n natural numbers.

$$\text{Let } S = 1^2 + 2^2 + 3^2 + \dots + n^2.$$

$$\text{We know that } x^3 - (x-1)^3 = 3x^2 - 3x + 1.$$

In this identity replacing x by $n, n-1, n-2, \dots, 3, 2, 1$ successively, we get

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$

$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$$

$$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$$

$$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$$

$\therefore$ adding up, we get

$$n^3 = 3[1^2 + 2^2 + 3^2 + \dots + n^2] - 3[1 + 2 + 3 + \dots + n] + n$$

$$\text{or } n^3 = 3S - 3 \frac{n(n+1)}{2} + n$$

$$\therefore 3S = n^3 - n + 3 \frac{n(n+1)}{2} = n(n+1)(n-1) + 3 \frac{n(n+1)}{2}$$

$$\therefore \begin{aligned} 6S &= 2n(n+1)(n-1) + 3n(n+1) \\ &= n(n+1)(2n-2+3) \\ &= n(n+1)(2n+1) \end{aligned}$$

$$\therefore S = \frac{n(n+1)(2n+1)}{6}.$$

Thus the sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

Note:—Using the sigma notation, the above may be expressed as $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$.

7.5. To find the sum of the cubes of the first n natural numbers.

Let S denote the required sum.

$$\text{Now } x^4 - (x-1)^4 = 4x^3 - 6x^2 + 4x - 1.$$

Replacing x in this identity by $n, n-1, n-2, \dots, 3, 2$ and 1 successively, we get

$$\begin{aligned}
 n^4 - (n-1)^4 &= 4n^3 - 6n^2 + 4n - 1 \\
 (n-1)^4 - (n-2)^4 &= 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1 \\
 (n-2)^4 - (n-3)^4 &= 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1 \\
 &\dots\dots\dots \\
 3^4 - 2^4 &= 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1 \\
 2^4 - 1^4 &= 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1 \\
 1^4 - 0^4 &= 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1
 \end{aligned}$$

Hence adding up, we get

$$n^4 = 4(1^3 + 2^3 + \dots + n^3) - 6(1^2 + 2^2 + \dots + n^2) + 4(1 + 2 + \dots + n) - n$$

$$\text{or } n^4 = 4S - 6 \frac{n(n+1)(2n+1)}{6} + 4 \cdot \frac{n(n+1)}{2} - n$$

$$\begin{aligned}
 \therefore 4S &= n^4 + n + n(n+1)(2n+1) + 2n(n+1) \\
 &= n(n+1)(n^2 - n + 1) + n(n+1)(2n+1) - 2n(n+1) \\
 &= n(n+1)(n^2 - n + 1 + 2n + 1 - 2) \\
 &= n(n+1)(n^2 + n) \\
 &= n^2(n+1)^2
 \end{aligned}$$

$$\therefore S = \left(\frac{n}{2}(n+1) \right)^2$$

Note :—This may be expressed as $\Sigma n^3 = \left[\frac{n(n+1)}{2} \right]^2$.

Thus the sum of the cubes of the first n natural numbers is $\left[\frac{n(n+1)}{2} \right]^2$.

This shows that the sum of the cubes of the first n natural numbers is a perfect square, namely, the square of the sum of the first n natural numbers.

EXERCISE XXXII A

Find the sum of the series :

1. $1^2 + 3^2 + 5^2 + \dots$ to n terms. (K. U. 1951)
2. $2^2 + 4^2 + 6^2 + \dots$ to n terms.
3. $1^3 + 3^3 + 5^3 + \dots$ to n terms.
4. $2^3 + 4^3 + 6^3 + \dots$ to n terms.
5. $(21)^2 + (22)^2 + \dots + (40)^2$
6. $(r+1)^2 + (r+2)^2 + \dots + n^2$ it being given that r and n are positive integers.

7. Show that the sum of the cubes of any number of consecutive integers (not necessarily beginning with unity) is divisible by the sum of the integers.

7.6. To find the sum of n terms of a series whose n th term is $an^3 + bn^2 + cn + d$.

Let S denote the sum.

nth term =	a	n^3	$+bn^2$	$+cn$	$+d$
(n-1)th term =	a	$(n-1)^3$	$+b(n-1)^2$	$+c(n-1)$	$+d$
(n-2)th term =	a	$(n-2)^3$	$+b(n-2)^2$	$+c(n-2)$	$+d$
.....					
3rd term =	a	3^3	$+b.3^2$	$+c.3$	$+d$
2nd term =	a	2^3	$+b.2^2$	$+c.2$	$+d$
1st term =	a	1^3	$+b.1^2$	$+c.1$	$+d$

$\therefore$ Summing up, we get

$$S_n = a(1^3 + 2^3 + \dots + n^3) + b(1^2 + 2^2 + \dots + n^2) + c(1 + 2 + \dots + n) + d \times n.$$

$$= a \left(\frac{n(n+1)}{2} \right)^2 + b \frac{n(n+1)(2n+1)}{6} + \frac{c(n+1)n}{2} + dn.$$

Note.—If the Sigma notation be used, then the result can be written down at once as shown below:

Let S_n denote the sum and T_n the n th term.

$$\begin{aligned} \text{Then } S_n &= \sum T_n = \sum (an^3 + bn^2 + cn + d) \\ &= a \sum n^3 + b \sum n^2 + c \sum n + d \sum 1 \end{aligned}$$

$$= a \left(\frac{n(n+1)}{2} \right)^2 + b \frac{n(n+1)(2n+1)}{6} + c \frac{n(n+1)}{2} + dn.$$

7.7. Miscellaneous Examples.

Ex. 1. Sum to n terms the series
 $3.5 + 4.7 + 5.9 + \dots$

Here each term is the product of two factors.

The factors 3, 4, 5..... are in A. P. having 3 as the first term and 1 as the common difference and therefore the n th factor is $3 + (n-1) \cdot 1$, i.e., $2+n$. Also the other factors 5, 7, 9 etc., are in A. P. having 5 as the first term and 2 as the common difference, and the n th factor is $5 + (n-1)2$ or $2n+3$.

$\therefore$ n th term of the given series is $(2+n)(2n+3)$

$$\text{or } n\text{th term} = 2n^2 + 7n + 6 \quad (1)$$

$$(n-1)\text{th term} = 2(n-1)^2 + 7(n-1) + 6$$

.....

$$\begin{array}{rcl} \text{2nd term} & = 2 \cdot 2^2 & + 7 \cdot 2 & + 6 \\ \text{1st term} & = 2 \cdot 1^2 & + 7 \cdot 1 & + 6 \end{array}$$

$\therefore$ adding up, we get

$$\begin{aligned} \text{Sum} &= 2(1^2 + 2^2 + \dots + n^2) + (1 + 2 + \dots + n) + 6n \\ &= \frac{2n(n+1)(2n+1)}{6} + \frac{7n(n+1)}{2} + 6n \\ &= \frac{2n(n+1)(2n+1) + 21n(n+1) + 36n}{6} \\ &= \frac{4n^3 + 27n^2 + 59n}{6} \end{aligned}$$

Note. Using sigma notation, we have from (1)

$$\text{Sum} = 2\Sigma n^2 + 7\Sigma n + 6\Sigma 1.$$

which is the same as (2).

Ex 2. Find the n th term and sum to n terms of the series
 $3 + 5 + 9 + 15 + 23 + \dots$

Let S denote the sum of n terms and T_n the general or n th term. Then

$$S = 3 + 5 + 9 + 15 + \dots + T_{n-1} + T_n$$

$$\text{Also } S = 3 + 5 + 9 + \dots + T_{n-1} + T_n$$

$$\begin{aligned} \therefore \text{Subtracting, } 0 &= 3 + 2 + 4 + 6 + \dots - T_n \\ &= 3 + (2 + 4 + 6 + 8 + \dots \text{to } n-1 \text{ terms}) - T_n \end{aligned}$$

$$\therefore T_n = 3 + \frac{n-1}{2} (4 + (n-2)2) = 3 + (n-1)n$$

$$\text{or } T_n = 3 - n + n^2$$

$$\therefore T_{n-1} = 3 - (n-1) + (n-1)^2$$

$$T_2 = 3 - 2 + 2^2$$

$$T_1 = 3 - 1 + 1^2$$

$$\begin{aligned} \therefore S &= 3n - (1 + 2 + \dots + n) + (1^2 + 2^2 + 3^2 + \dots + n^2) \\ &= 3n - \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{2n^3 + 16n}{6} = \frac{n^3 + 8n}{3} \end{aligned}$$

Note. The peculiarity in this series is that the series formed by the differences $T_2 - T_1$, $T_3 - T_2$, $T_4 - T_3$ and so on is an A. P. All such series can be summed by the above method. If the above differences form a G. P., a similar method gives the result.

Ex 3. Sum up the series $1+5+13+29+\dots$ to n terms.

Let S denote the sum T_n the n th term

$$\text{Then } S = 1 + 5 + 13 + 29 + \dots + T_{n-1} + T_n$$

$$\text{Also } S = 1 + 5 + 13 + \dots + T_{n-1} + T_n$$

$\therefore$ Subtracting, $0 = 1 + 4 + 8 + 16 + \dots$ to n terms $- T_n$

$\therefore T_n = 1 + (4 + 8 + 16 + \dots$ to $n-1$ terms)

$$= 1 + \frac{4(2^{n-1}-1)}{2-1} = 1 + 4(2^{n-1}-1)$$

$$\text{or } T_n = 4 \cdot 2^{n-1} - 3$$

$$\therefore T_{n-1} = 4 \cdot 2^{n-2} - 3$$

$$T_2 = 4 \cdot 2^1 - 3$$

$$T_1 = 4 \cdot 2^0 - 3$$

$\therefore$ adding up, we get

$$S = 4(1 + 2 + 2^2 + 2^3 + \dots + 2^{n-2} + 2^{n-1}) - 3n$$

$$= 4 \cdot \frac{(2^n - 1)}{2 - 1} - 3n = 4 \cdot 2^n - 4 - 3n$$

$$= 2^{n+2} - 3n - 4.$$

Ex. 4. Sum up $5+55+555+\dots$ to n terms.

Here $S = 5 + 55 + 555 + \dots + T_{n-1} + T_n$

Also $S = 5 + 55 + \dots + T_{n-1} + T_n$

$\therefore 0 = 5 + 50 + 500 + \dots - T_n$

$\therefore T_n = 5 + 50 + 500 + \dots$ to n terms

$$= \frac{5(10^n - 1)}{10 - 1} = \frac{5(10^n - 1)}{9}$$

$$\text{or } T_n = \frac{5}{9} (10^n - 1)$$

$$\therefore T_{n-1} = \frac{5}{9} (10^{n-1} - 1)$$

$\dots$

$$T_2 = \frac{5}{9} (10^2 - 1)$$

$$T_1 = \frac{5}{9} (10 - 1)$$

$$\therefore \text{Sum} = \frac{5}{9} \{10 + 10^2 + \dots + 10^n - n\}$$

$$= \frac{5}{9} \left\{ \frac{10(10^n - 1)}{10 - 1} - n \right\}$$

$$= \frac{5}{9} \left\{ \frac{10(10^n - 1)}{9} - n \right\}.$$

Note.—The n th term is $555\dots 5$ having n digits $= 5 + 50 + 500 + \dots$ to n terms, a fact which can also be used to find T_n in the above form.

Ex. 9. Find the sum of the series

$\frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} \dots$ to n terms; and find sum to infinity.

Let S_n be the sum and T_n the n th term, then

$$T_n = \frac{1}{[3 + (n-1)4][7 + (n-1)4]} = \frac{1}{(4n-1)(4n+3)}$$

$$= \frac{1}{4} \left(\frac{1}{4n-1} - \frac{1}{4n+3} \right).$$

Giving to n ; values, 1, 2, 3.....we get

$$T_1 = \frac{1}{4} \left[\frac{1}{3} - \frac{1}{7} \right]$$

$$T_2 = \frac{1}{4} \left[\frac{1}{7} - \frac{1}{11} \right]$$

$$T_3 = \frac{1}{4} \left[\frac{1}{11} - \frac{1}{15} \right]$$

.....

$$T_{n-1} = \frac{1}{4} \left[\frac{1}{4n-5} - \frac{1}{4n-1} \right]$$

$$T_n = \frac{1}{4} \left[\frac{1}{4n-1} - \frac{1}{4n+3} \right]$$

Adding, we have

$$S_n = \frac{1}{4} \left[\frac{1}{3} - \frac{1}{4n+3} \right] = \frac{n}{3(4n+3)} = \frac{1}{12 + \frac{9}{n}}.$$

Note.—When $n \rightarrow \infty$, $S \rightarrow \frac{1}{12}$.

EXERCISE XXXII B

- Find the sum to n terms of the series whose n th term is (i) $n(n+2)$. (ii) $3n^2+2n$. (iii) $4n^3+6n^2+2n$.

Find the sum of the series :

- $3 \times 5 + 5 \times 7 + 7 \times 9 + \dots$ to p terms.

- Find the n th term and the sum to n terms of the series $1.2 + 2.3 + 3.4 + \dots$

- Sum up to n terms the series $1.2^2 + 2.3^2 + 3.4^2 + \dots$

5. Sum up $1 + (1+2) + (1+2+3) + \dots + (1+2+3+\dots+n)$.
6. Find the sum to n terms of
- (i) $1 + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2} + \frac{1}{4}\right) + \dots$
- (ii) $1 + \left(1 + \frac{1}{3}\right) + \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) + \dots$
7. Sum up to n terms the series whose n th term is $2^n - 1$.
8. Sum up $1 + 4 + 8 + 13 + \dots$ to n terms.
9. Sum up $1 + 3 + 6 + 10 + 15 + \dots$ to n terms.
10. Find the sum to n terms of the series
 $1 + 5 + 12 + 22 + 35 + \dots$
11. Find the n th term and the sum of n terms of the series
 $3 + 7 + 14 + 24 + 37 + \dots$ (P. U. 1946)
12. Sum up $3 + 5 + 11 + 20 + \dots$ to n terms.
13. Sum up $1 + 11 + 111 + \dots$ to n terms.
14. Sum up to n terms the series $7 + 77 + 777 + \dots$
15. Sum to n terms the series
 $1 + 3 + 7 + 15 + 31 + \dots$
16. (i) $3 + 6 + 11 + 20 + 37 + \dots$ to n terms (Q. 1948)
(ii) $2 + 7 + 14 + 23 + \dots$ to n terms (K. U. 1949)
17. $1 + 4 + 13 + 40 + 121 + \dots$ to n terms. (Q. 1944)
18. Find the sum to n terms of the series
 $1.2.3 + 2.3.4 + 3.4.5 + \dots$ (P. U. 1943)
19. Find the n th term and sum to n terms of
- (i) $1.2.3 + 2.3.5 + 3.4.7 + 4.5.9 + \dots$ (Q. 1946)
- (ii) $-6 - 4 + 0 + 8 + 24 + \dots$ (Q. 1945)
20. Sum up to n terms $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$
21. Find the n term and the sum to n terms of the series
 $1.3.6 + 2.4.8 + 3.5.10 + \dots$
22. Sum to n terms the series $2.3.1 + 3.4.2 + 4.5.3 + \dots$
23. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots$
24. $\frac{1}{5.8} + \frac{1}{8.11} + \frac{1}{11.14} + \dots$
25. $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots$
26. Find the n th term and sum to n terms of
 $2 + 4 + \frac{14}{3} + \frac{44}{9} + \frac{134}{27} + \dots$

7.8. Files of Shot :

To find the number of shot in a complete pyramidal pile of n layers the base being (i) an equilateral triangle (ii) a square (iii) a rectangle.

(i) **Triangular Pile.** Since there is only one shot on the top of the pyramid and since the number of shot in each side of a particular layer will be one less than that a layer immediately below, it is clear that each side of the triangular base will contain n shot.

Thus number in the bottom (n th layer) is

$$= n + (n-1) + (n-2) + \dots + 1$$

$$= \frac{n(n+1)}{2}$$

$$\therefore \text{Total number in all the layers} = \sum \left(\frac{n(n+1)}{2} \right) = \frac{1}{2} \sum n^2 + \frac{1}{2} \sum n$$

$$= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{1}{6} n(n+1)(n+2).$$

(ii) **Square Pile.** Here the number of shot in bottom layer $= n^2$

$$\therefore \text{Total number} = \sum n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(iii) **Rectangular Pile.** Let m be the number of shot in length of the lowest rectangle and n the number in breadth, each successive layer will contain one less in breadth and one less in length and there being n layers breadth being reduced to 1, in the top most layer.

The number of shot in the bottom layer $= mn$

$$\text{,, ,, next ,, ,,} = (m-1)(n-1)$$

.....

$$\text{,, ,, top ,, ,,} = (m-n+1).1.$$

$$\therefore \text{Total number} = (m-n+1).1 + (m-n+2).2$$

$$+ (m-n+3).3 + \dots \text{to } n \text{ terms.}$$

$$= (m-n)(1+2+3+\dots+n) + (1^2+2^2+\dots+n^2)$$

$$= \frac{1}{6} n(n+1)(3m-n+1).$$

EXERCISE XXXIII

1. The number of shot in a complete square pile is to the number of shot in a complete triangular pile of 4 times the number of layers as 1 to 24. Find the number of shot in each pile. (M. U.)

2. Find the number of shot in a complete square pile having 15 shot in each side of the base.

3. Find the number of shot in a complete triangular pile, each side of the base containing 10 shot.

4. Find the number of shot in a complete square pile having 625 shot in its base.

5. Find the number of shot in a complete rectangular pile, whose base is 12×8 .

Notes on Induction

Induction is one of very useful methods employed in Algebra to prove a certain given result. The method will be best illustrated and explained by the following example :

Show that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$, for all positive integral values of n .

Assume the formula to be true for some definite positive integral number m , i.e., assume that

$$1+2+3+4+\dots+m=\frac{m(m+1)}{2}. \quad \dots(1)$$

Then adding $(m+1)$ to both the sides we get

$$1+2+3+\dots+m+(m+1)=\frac{m(m+1)}{2}+m+1=\frac{m^2+3m+2}{2}$$

$$\text{i.e., } 1+2+3+4+\dots+m+(m+1)=\frac{(m+1)(m+2)}{2},$$

which is exactly of the same form as (1) except that m is changed to $(m+1)$.

So that we find that if the result is true for some definite positive integral number m , then it is true for $(m+1)$ also. $\dots(2)$

But $1+2=3=\frac{2(2+1)}{2}$, and thus the formula is true for $n=2$.

Hence, since the formula is true for $n=2$, therefore by (2) the formula is true for $n=3$, and therefore by (2) the formula is true for $n=4$, and so on. Hence the formula is generally true for all positive integral values of n .

Note.—There are three steps in the above method :

(1) To show that if the result is true for some positive integral number m , then it is also true for the next integer $m+1$. (2) To show that the result is true for some number n say, $n=1$ or 2 (3) Then to generalise the result for all positive integral values of n .

Note 2.—Observe that the method of Induction is applicable only when n is a positive integer.

We shall use this method later on in proving the Binomial Theorem.

Ex. Prove the following by the method of Induction :

(i) $1+3+5+\dots+(2n-1)=n^2$.

(ii) $1^2+2^2+3^2+\dots+n^2=\frac{1}{6}n(n+1)(2n+1)$

(iii) $1^3+2^3+3^3+\dots+n^3=\left[\frac{n(n+1)}{2}\right]^2$

(iv) $a+ar+ar^2+\dots+ar^{n-1}=\frac{ar^n-a}{r-1}$.

(v) x^n-y^n is divisible by $x-y$ for all positive integral values of n .

REVISION QUESTIONS II

1. Sum to n terms the series

$$(3a-2b)+(4a-5b)+(5a-8b)+\dots$$

2. Find the n th term of the series

$$\frac{1}{n}+\frac{n+1}{n}+\frac{2n+1}{n}+\dots$$

3. Sum up to n terms the series

(i) $0.7+0.71+0.72+0.73+\dots$

(ii) $\left(1-\frac{1}{n}\right)+\left(1-\frac{2}{n}\right)+\left(1-\frac{3}{n}\right)+\dots$

(iii) $(n-1)+(n-2)+(n-3)+\dots$

4. The sum of n terms of an A.P., is 40, the common difference is 2 and the last term is 13 ; find n .

5. If $2n+1$ terms of the series

$$1+3+5+7+9+\dots$$

be taken prove that the sum of the alternate terms, namely,

$$1+5+9+\dots$$

will be to the sum of the remaining terms as $n+1$ is to n .

6. There are four numbers in A.P., the sum of the two extreme numbers is 8 and the product of the two mean ones is 15. Find the numbers.

7. Find four numbers in G. P. so that their sum may be 15 and the sum of their squares 85.

8. The p th term of an A. P. is $\frac{1}{q}$, and the q th term is $\frac{1}{p}$. Show that the sum of pq terms is $\frac{1}{2}(pq+1)$.

9. If $(q-r)^2$, $(r-p)^2$, $(p-q)^2$, are in A. P., show that $\frac{1}{q-r}$, $\frac{1}{r-p}$, $\frac{1}{p-q}$ are in A.P.

10. The interior angles of a rectilinear figure are in A. P., the least is 120° and the common difference is 5° . Find the number of sides.

11. There are n arithmetic means between 3 and 54 such that the 8th mean is to the $(n-2)$ th mean as 3 is to 5; find n .

12. Sum up to infinity the series

$$(i) \sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \dots (ii) 1+x+x^2+\dots$$

$$(iii) \frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \dots (iv) 2+2+\frac{3}{2}+1+\frac{5}{8}+\dots$$

13. Show that $2^{\frac{1}{4}}, 4^{\frac{1}{8}}, 8^{\frac{1}{16}}, 16^{\frac{1}{32}}, \dots$ to $\infty = 2$.

14. The sum of two quantities is n times their geometric mean. Compare the two quantities.

15. In an infinite geometric series, each term is equal to three times the sum of all terms which follow it, and the sum of the first two terms is 15. Find the sum to infinity of series.

(B. U.)

16. If a, b, c and d are in G. P., prove that

$$(i) (b+c)(b+d) = (c+a)(c+d).$$

$$(ii) a^2-b^2, b^2-c^2 \text{ and } c^2-d^2 \text{ are in G. P.}$$

17. If a be the sum, b the product and c the sum of the reciprocals of n terms in G. P., prove that

$$b^2 = \left(\frac{a}{c} \right)^n.$$

18. Find the n th term of

$$(i) \quad 4 + 4\frac{2}{7} + 4\frac{8}{13} + 5 + \dots$$

$$(ii) \quad 2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \dots$$

19. If the m th term of a series in H. P. be n and n th term be m , prove that the r th term is $\frac{mn}{r}$.

20. An A. P. and an H. P. have the same first term, the same last term and the same number of terms; prove that the product of the r th term from the beginning in one series and the r th term from the end in the other is independent of r .

21. If P , Q and R are the p th, q th and r th terms of an A.P., prove that

$$P(q-r) + Q(r-p) + R(p-q) = 0.$$

22. Sum to n terms of the series

$$x + 2\left(x - \frac{1}{n-1}\right) + 3\left(x - \frac{2}{n-1}\right) + 4\left(x - \frac{3}{n-1}\right) + \dots \quad (\text{P. U.})$$

23. If m be the arithmetic mean of n consecutive integers, show that the sum of their cubes is

$$mn\left[m^2 + \frac{n-1}{4}\right]. \quad (\text{P. U. 1927}).$$

24. Sum up the series :

$$(i) \quad 2 + 5 + 10 + 17 + 26 + \dots \text{ to } n \text{ terms.}$$

$$(ii) \quad 1 + 3 + 7 + 15 + \dots \text{ to } n \text{ terms.}$$

$$(iii) \quad 1 + 2x + 3x^2 + 4x^3 + \dots \text{ to } 100 \text{ terms.}$$

25. (i) $1 - x + 2x^2 - 3x^3 + 4x^4 - \dots$ to infinity.

$$(ii) \quad 1 + 3x + 4x^2 + 5x^3 + \dots \text{ to infinity.}$$

$$(iii) \quad 1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2.$$

26. Evaluate : $1 + 4x + 9x^2 + 16x^3 + 25x^4 + \dots$ to infinity for

$$x = \frac{1}{5}.$$

27. Sum up to n terms the series
 $a - (a + d) + (a + 2d) - (a + 3d) + \dots$

28. Sum up to n terms the series
 $1 - 3 + 6 - 10 + 15 - 21 + \dots$

29. Sum up the series $1^3 + 2^3 + 3^3 + \dots + n^3$.

30. Sum the series

$$n.1 + (n-1)2 + (n-2)3 + (n-3)4 + \dots + 1.n.$$

[Hint : r th term $= \{n - (r-1)\}r = (n+1)r - r^2$.]

31. If a be the first term and l the last term of an A. P., and if an arithmetic mean be inserted between every two successive terms of the series, prove that the sum of the original series exceeds the sum of the arithmetic means by $\frac{1}{2}(a+l)$.

32. Show that the sum of

$$1 + 2^2 + 3 + 4^2 + 5 + 6^2 + \dots$$

to n terms is $\frac{1}{12}(n+1)(2n^2+n+3)$, or $\frac{1}{12}n(2n^2+9n+4)$,

according as n is odd or even.

33. If $S_1, S_2, S_3, \dots, S_n$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \dots, n$ and whose common ratios are

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+1} \text{ respectively.}$$

prove that $S_1 + S_2 + S_3 + \dots + S_n = \frac{1}{2}n(n+3)$.

34. If S_p represents the sum of p terms of a G. P., whose first term is a and common ratio r , find $S_1 + S_2 + S_3 + \dots + S_p$.

35. If the sum of n terms of a series be $a(r^n - 1)$, prove that the series is a G. P., and find the common ratio.

36. Show that if one geometric mean a and two arithmetic means, b and c be inserted between two numbers x and y , then :

$$a^2 = (2b - c)(2c - b).$$

37. If one arithmetic mean a and two geometric means, b and c , be inserted between any two numbers, then

$$b^3 + c^3 = 2abc.$$

38. Prove that the sum of the series $a + b + \dots + l$

(i) when it is a G. P. is $\frac{bl - a^2}{b - a}$.

(ii) when it is an A. P. is $\frac{(a+l)(l+b-2a)}{2(b-a)}$.

39. The series of natural numbers are written as follows :—

1
2 3
4 5 6
7 8 9 10
.....

Prove that the sum of the numbers in the n th row is $\frac{1}{2}n(n^2+1)$.

40. If S_1, S_2, S_3 denote the sums of n terms of three arithmetic series whose first terms are unity and whose common differences are d_1, d_2, d_3 , respectively, show that

$$\frac{S_1-n}{d_1} = \frac{S_2-n}{d_2} = \frac{S_3-n}{d_3}.$$

41. If $S_1, S_2, S_3, \dots, S_p$ are the sums to n terms of p arithmetic series whose first terms are $1, 2, 3, \dots$ and common differences are $1, 3, 5, \dots$ show that

$$S_1 + S_2 + \dots + S_p = \frac{1}{2}pn(pn+1).$$

42. Find the values of n for which the expression $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ will be the A. M., G. M., and H. M. respectively between a and b . (P. U.)

43. Prove that; (i) the A. M. between two unequal quantities is greater than their G. M. (ii) the G. M. between the same quantities is also the G. M. of their A. M. and H. M. (iii) if a, b, c are in A. P. and b, c, d in H. P. then will $a : b = c : d$. (P. U.)

44. If the number of balls in an complete square pile is 3311, how many layers are there?

45. Find the number of shot in a unfinished triangular pile having 25 shot in each side of the bottom layer and 18 in each side of the top layer.

46. Show that the sum of the product of the first n natural numbers taken two at a time is

$$\frac{n}{24}(n^2-1)(3n+2).$$

CHAPTER VIII

PERMUTATIONS AND COMBINATIONS

PERMUTATIONS

8.1. Problems similar to the following must have occurred to the student on many occasions :

There are four doors to a lecture room ; in how many ways can a student enter the room and leave it by a different door ?

Let us denote the doors by the numbers 1, 2, 3, 4. The various ways of entering and leaving the room are shown below :

E		1		2		3		4
L		2,3,4		1,3,4,		1,2 4,		1,2,3

It is evident that there are four ways of entering the room and that when this has been done by one of the doors, the room may be left by any one of the remaining three doors, i.e., there are three ways of leaving the room corresponding to every one of the four ways of entering it. Thus there are $4 \times 3 = 12$ ways of entering and leaving room.

8.2. The above problem leads to the following principle :—

If one operation can be performed in m ways and if corresponding to each of these m ways of performing this operation, there are n ways of performing a second operation, then the number of ways of performing the two operations together is $m \times n$.

Examples

1 There are 8 buses running between Delhi and Mehrauli, in how many ways can a man go from Delhi to Mehrauli and return by a different bus.

There are 8 ways of making the first journey and for each of these 8 ways, there are seven ways of returning to Delhi because he is to return by different bus & not-by the same bus. Hence the number of ways of making the two journeys is $8 \times 7 = 56$.

2. A gentleman has got 6 sorts of note papers, 7 different pens and 4 different ink-stands ; show that he can begin to write a letter in 168 ways.

3. A student has six shirts and ten ties. How many different combinations of shirt and tie can he wear. Ans. 60.

4. There are 5 ways from A to B and 3 from B to C ; how many ways are there from A to C *via* B ? Ans. 15.

5. In a crossword puzzle there are two solutions to three places and three solutions to one other place. How many different solutions could be sent in ? Ans. 24.

8.21. We now take a few other similar problems.

Ex. 1. How many numbers of two digits can be formed out of the four digits 1, 2, 3, 4 ?

To form numbers of two digits, we have to fill up two places, the ten's and unit's.

There are 4 ways of filling the ten's place ; for any one of the digits 1, 2, 3, 4, may be put into it. When the ten's place has been filled up in any one of these 4 ways, three digits are left and the unit's place can be filled up in three ways. Thus there are 3 ways of filling the unit's place corresponding to every one of the 4 ways of filling the ten's place. Hence the two places can be filled up in $4 \times 3 = 12$ ways and we get 12 numbers which are as follows :

12	21	31	41
13	23	32	42
14	24	34	43

Ex. 2. Five subjects are taught to a class ; in how many ways can the time-table of the first three periods be formed ?

Any one of the five subjects may be taken up in the first period, i.e., there are five ways of filling the first period. When this has been done, 4 subjects are left and the second period can be filled up in 4 ways, i.e., for each of the five ways of

filling the first period, there are 4 ways of filling the second and thus there are $5 \times 4 = 20$ ways of filling the first two periods.

Now there are three subjects left and so there are three ways of filling the third period corresponding to every one of the $5 \times 4 = 20$ ways of filling the first two periods. Hence altogether there will be $20 \times 3 = 60$ ways of filling the three periods.

Ex. 3. In how many ways can four books on different subjects be arranged on a shelf?

Any one of the 4 books may occupy the first place on the shelf; 3 books are left and so there are 3 ways of filling the second place for each of the 4 ways of filling the first place.

The first two places can, therefore, be filled up in $4 \times 3 = 12$ ways. When this has been done, two books are left and any one of them may be placed in the third place, i.e. there are two ways of filling the third place for every one of the 12 ways of filling the first two places.

The one book which is left will occupy the last place. Hence there are $12 \times 2 = 24$ ways of arranging 4 books on the shelf.

8.3. In the above examples we have obtained *all possible arrangements of a number of things taken some or all of them at a time*. These arrangements are called **Permutations**.

We now follow the same argument in dealing with the more general case of finding the number of permutations of n different things taken r of them at a time. For the sake of brevity this number is denoted by ${}^n P_r$.

8.31. To find the number of permutations of n things taken r of them at a time.

Case I :—When all the n things are different and none of them is to be repeated in any arrangement.

The number of permutations of n things taken r at a time is the same as the number of ways of filling r places out of the n given things. ($r < \text{or} = n$).

So that n different things are at our disposal and we have to fill up r places with them.

The first place can be filled by any one of the n things, i.e., there are n ways of filling up the first place.

When the first place has been filled up in one of the n ways, $n-1$ things are left and any one of them can be placed in the second place, so that there are $n-1$ ways of filling up the second place for *each* of the n ways of filling the first place.

When the first two places have been filled up in one of the $n(n-1)$ ways, there are $(n-2)$ things left and one of them can be placed in the third place, so that there are $n-2$ ways of filling up the third place corresponding to *every one* of the $n(n-1)$ ways of filling up the first two places.

There are, therefore, $n(n-1)(n-2)$ ways of filling up the first three places.

Proceeding in this way, we notice that a new factor is introduced each time we fill a place and that at any stage the number of factors thus obtained is the same as the number of places filled up.

Hence the number of ways in which r places can be filled up

$$\begin{aligned}
 &= n(n-1)(n-2)\dots\dots\dots\text{up to } r \text{ factors} \\
 &= n(n-1)(n-2)\dots\dots\dots[n-(r-1)] \\
 &= n(n-1)(n-2)\dots\dots\dots(n-r+1).
 \end{aligned}$$

Thus the number of permutations of n different 'things' taken r at a time is given by

$${}^n P_r = n(n-1)(n-2)\dots\dots(n-r+1).$$

Cor. 1. The number of permutations of n different things taken all of them at a time is

$$\begin{aligned}
 {}^n P_n &= n(n-1)(n-2)\dots\dots n \text{ factors} \\
 &= n(n-1)(n-2)\dots\dots 3.2.1.
 \end{aligned}$$

= product of all natural numbers from 1 to n .

Such a product is denoted by the symbol $|n$ or $n!$ and read as "factorial n ".

$$\therefore {}^n P_n = |n.$$

$$\begin{aligned}\text{Cor. 2. } {}^nP_r &= n(n-1)(n-2)\dots\dots(n-r+1) \\ &= \frac{n(n-1)\dots (n-r+1)(n-r)\dots\dots 3.2.1.}{(n-r)\dots\dots 3.2.1.}\end{aligned}$$

$$\therefore {}^nP_r = \frac{n!}{(n-r)!}$$

Examples

Ex. 1. Show that :

$$(i) {}_{23}P_{22} = {}_{21}P_{23} \quad (ii) {}_nP_{n-1} = n!$$

$$(iii) n(n-1)(n-2)\dots(n-r+1) = {}_nP_r$$

$$(iv) {}^7P_3 = 7 \times 6 \times 5. \quad (v) {}^{14}P_9 = \frac{14!}{5!}$$

$$(vi) {}^nP_n = n \times {}^{n-1}P_{n-1}.$$

Ex. 2. Write down all the permutations of the letters a, b, c, d , taken two at a time.

Here we are to find the number of permutations of four different letters taken two at a time, therefore, their number $= {}^4P_2 = 4 \times 3 = 12$.

These Permutations are :

ab	ba	ca	da
ac	bc	cb	db
ad	bd	cd	dc

Ex. 3. Find the numbers of words that can be formed of the letters a, b, c, d taken two or more at a time.

Here if 2 letters out of 4 are taken the number of words $= {}^4P_2 = 12$,

if 3 out of 4 are taken no. of words $= {}^4P_3 = 24$;

if all the four are taken no. of words $= {}^4P_4 = 24$;

$\therefore$ Total no. of words formed $= 12 + 24 + 24 = 60$.

Ex. 4. Find no. of words that can be formed of the letters a, b, c, d ,

Here the no. of one letter words $= {}^4P_1 = 4$
 „ „ two „ „ $= {}^4P_2 = 12$
 „ „ three „ „ $= {}^4P_3 = 24$
 „ „ four „ „ $= {}^4P_4 = 24$
 Total $= 24 + 24 + 12 + 4 = 64$.

Ex 5. In how many ways can the letters of the word SOCIAL be arranged? How many of these arrangements, will begin with S? How many will not begin with S? How many will begin with S and end with L?

$\therefore$ the required number of ways,
 $= |6| = 720$.

Now if S remains fixed we are left with five letters which can be arranged among themselves in 5. i.e. 120 ways.

The number of words which do not begin with S = the total number – the words which begin with S
 $= 720 - 120 = 600$.

Again if the places of S and L remain fixed then the remaining letters to be permuted are four.

Here total number of letters in the word SOCIAL is six.

$\therefore$ the required number of ways $= |4| = 24$.

Ex. 6. In the above example, fixed the number of words which can be formed if O and C may always come together.

Here O and C are to come together which means that these two can be counted as one letter, so that the five letters S(OC) I, A and L can be arranged in 5 ways. Moreover O and C can be arranged between themselves in two ways, viz., OC and CO.

$\therefore$ the total number of words that can be formed are
 $2|5| = 2 \times 5 \times 4 \times 3 \times 2 \times 1$
 $= 240$.

Ex. 7. How many numbers of six digits can be formed from the digits 4, 5, 6, 7, 8, 9, no digit being repeated? How many of them are not divisible by 5?

The number of numbers which can be formed with the 6 digits
 $= {}^6P_6 = |6|$.

out of these 6 numbers those which have 5 in the units places are divisible by 5 and their number = 5, obtained by arranging the remaining 5 figures among themselves.

∴ The numbers not divisible by 5 are

$$= \underline{6} - \underline{5} = 6 \underline{5} - \underline{5} = 5 \underline{5} = 600.$$

Ex. 8. Find all the five digit numbers formed from the digits 0, 1, 2, 3, 4.

Here with 5 digits at our disposal we have to form numbers having 5 digits.

∴ number of such numbers = 5.

But when 0 occurs on the extreme left the number is reduced to 4 digit number. The number of such numbers = 4.

∴ All the five digit numbers are 5 - 4 = 96 in number.

Ex. 9. Find out the number of all possible numbers formed of the digits 4, 5, 6, 7, 8, 9.

Here number of :—

(i) one digit numbers	$= {}^6P_1 = 6$
(ii) two digit " "	$= {}^6P_2 = 30$
(iii) three " "	$= {}^6P_3 = 120$
(iv) four " "	$= {}^6P_4 = 360$
(v) five " "	$= {}^6P_5 = 720$
(vi) six " "	$= {}^6P_6 = 720$

Total = 1956.

Ex. 10. Find the number of signals which can be given with six flags of different colours.

When one flag is used, the No. of Sig. = ${}^6P_1 = 6$.

When two flags are used, the No. of Sig = ${}^6P_2 = 6 \times 5 = 30$.

" three " " " " " $= {}^6P_3 = 6 \times 5 \times 4 = 120$.

" four " " " " " $= {}^6P_4 = 6 \times 5 \times 4 \times 3 = 120$.

" five " " " " " $= {}^6P_5 = 6.5.4.3.2 = 720$.

" all " " " " " $= {}^6P_6 = \underline{6} = 720$.

Hence the required number of signals

$$= {}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6$$

$$= 6 + 30 + 120 + 360 + 720 + 720 = 1956.$$

Ex. 11. How many of the numbers formed in Ex. 9 above are divisible by 2 ?

Those numbers in Ex. 9 above are divisible by 2 which will have 4, or 6, or 8 in the units place.

$\therefore$ number of one digit numbers divisible by 2 = 3

"	"	two	"	"	"	$2 = {}^5P_1 \times 3 = 15.$
"	"	three	"	"	"	$2 = {}^5P_2 \times 3 = 60.$
"	"	four	"	"	"	$2 = {}^5P_3 \times 3 = 180.$
"	"	five	"	"	"	$2 = {}^5P_4 \times 3 = 360.$
"	"	six	"	"	"	$2 = {}^5P_5 \times 3 = 360.$

Now add all.

EXERCISE XXXIV A

- Find the value of n where
(i) ${}^nP_2 = 42$ (ii) ${}^nP_3 = 20n$ (iii) ${}^nP_8 = 11 \cdot {}^nP_7$ (iv) ${}^nP_4 = 88 \times {}^{n-1}P_2$
- Show that (i) $\frac{{}^{2n}}{n} = 1.3.5 \dots (2n-1) \times 2^n$.
(ii) ${}^nP_r = n \cdot {}^{n-1}P_{r-1}$ (iii) ${}^nP_r = (n-r+1) {}^nP_{r-1}$.
- Prove that ${}^nP_r = {}^{n-1}P_r + {}^{n-1}P_{r-1}$.
- Write down all the arrangements of the letters a, b, c and d taken three of them at a time.
- Find the number of arrangements that can be made from the letters of the word DELHI.
(i) taking all together, (ii) taking three at a time.
(iii) beginning with D and ending in I.
(iv) the letter L always occupying the middle position.
- Write down all numbers of 4 digits formed with the digits 5, 7, 2 and 9.

How many are they in all ?

How many of them are more than 9000 ?

How many of them are even numbers ?

How many of them are divisible by 5 ?

7. How many numbers can be formed taking three (i) out of the five digits 3, 4, 6, 8, 9 (ii) taking all of them ?

8. How many numbers of 4 digits are possible out of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 ?

9. How many numbers are formed by using all the digits 2, 4, 5, 7, 9 no digit being repeated ?

(i) How many of these numbers are multiples of 5 ?

(ii) How many of these are even ?

(iii) How many of these are greater than 90,000 ?

(iv) How many of these are less than 50,000 ?

10. The figures 3, 4, 5, 6, 7, 8, 9 are written down in every possible order ; find how many of these numbers thus formed lie between 5,000,000 and 8,700,000.

11. How many three digit—numbers of the digits 0, 2, 5, 9 can be formed. Also find the number of those which are not divisible by 5.

12. Three persons go into a railway carriage, where there are six vacant seats. In how many different ways may they seat themselves ?

13. In how many ways can the letters of the word SPECTRUM be arranged ? What will be the number of ways if the first letter must be T ?

14. How many signals can be made with 5 different flags when any number of them may be hoisted at a time ?

15. In how many ways can the letters of the word PAYMENT be arranged so that the letters Y and M may be always together.

16. Seven candidates are to take examination, 2 in Mathematics and remaining in different subjects. In how many ways can they be seated in a row so that the two examinees in Mathematics may not sit together.

8.32. Case II. To find the number of permutations of n different things taken r at a time when each thing may be repeated any number of times in any arrangement.

Here we have to fill up r places out of n different things when each thing can be used as often as we please, in any arrangement.

Since any one of the n things can be placed in the first place, therefore there are n ways of filling the first place. When this has been done, there are again n ways of filling the second place corresponding to each of the n ways of filling the first ; for, the thing which occupies the first place can be repeated and placed in the second place also. Thus there are altogether $n \times n$, i.e., n^2 ways of filling the first two places.

The third place can also be filled in n ways and each way of filling the third place may be associated with each of the n^2 ways of filling the first two places. Therefore the first three places can be filled in $n^2 \times n$ i.e., n^3 ways.

Proceeding in this way we find that the index of n is the same as the number of places filled up.

Thus the number of ways of filling r places is $(n)^r$.

$\therefore$ the required number of permutations $= (n)^r$.

Ex. 1. Write down all the permutations of the letters a, b, c taken two at a time, when letters may be repeated.

Here $n=3$, $r=2$ therefore the required permutations are $3^2=9$ in number. These permutations are :—

aa	ba	ca
ab	bb	cb
ac	bc	cc

Ex. 2. A number-lock consists of 3 rings each marked with 6 different digits. How many times a thief will remain unsuccessful in opening it.

Here each of the three rings can be set in position in 6 ways and so the number of numbers which can be formed will be $(6)^3$ i.e. 216. But one of these is a code number for opening the lock. Therefore unsuccessful attempts will be 215.

Ex 3. In how many ways can 5 prizes be given away to 4 boys, when (i) each boy is eligible for all the prizes ; (ii) when a boy is not eligible for all the prizes.

There are 4 ways of giving away the first prize as it can be given to any one of the 4 boys ; there are again 4 ways of disposing of the second prize and so on the required number $= 4^5$.

(ii) Subtract 4 from 4^5 , 4 being the number in which a boy may have all the prizes. any one of them being given all the prizes.

EXERCISE XXXIV B

1. How many numbers of four digits can be formed, using all the digits 3, 4, 5 and 6?

(i) When no digit is to be repeated.

(ii) When digits may be repeated.

2. A letter-lock consists of 4 rings each marked with 5 different letters. Find the number of ways a code word be fixed to open it.

3. In how many ways may n prizes be given to n boys;

(i) so that each boy may have a prize.

(ii) when there is no restriction to the number of prizes boy receives.

(iii) when no boy should receive all the prizes.

4. Find the number of ways in which 6 different things can be divided among 4 boys when a boy may receive all the 6 things

5. In how many ways can 5 different hats be divided between 2 men.

6. How many numbers less than 1000 can be formed with digits 1, 2, 3, 4, 5, when digits may be repeated.

7. How many numbers less than 1000 can be formed with digits 0, 2, 3, 4, 5, when digits may be repeated.

(Here three digit numbers will be 5^3 including numbers when 0 occurs towards the left i.e., including numbers of two digits also.

Now (0, 0, 0) is no number. So that we reject the number in which all the digits are zeros.

$\therefore$ required number of numbers $= 5^3 - 1 = 124$.]

8.33. Case III. *To find the number of permutations of n things taken all together, when p of the things are alike and of one kind, q of them alike and of the other kind, and the rest all different.*

(The case when permutations are made by taking some and not all the given things will be discussed later vide Art 8.82.)

Let the n things be denoted by n letters; p of them being alike, each denoted by a , q of them being alike, each denoted by b and the rest being all unlike, say, c, d, e , etc.

Let x be the required number of permutations. In any one of these x permutations let us replace the p like letters a by unlike letters (say, a_1, a_2 , etc.) different from any of the rest and let us arrange these p new letters among themselves without altering the positions of any of the remaining letters. We thus form $|p|$ new permutations, and if the changes were made in each of the x permutations the total number of permutations formed will be $x \times |p|$.

In like manner, if the q letters b were also replaced by q unlike letters, the total number of permutations would be

$$x \times |p| \times |q|.$$

But the letters are now all different and n in number and therefore, they admit of $|n|$ permutations.

Hence
$$x \times |p| \times |q| = |n|$$

or
$$x = \frac{|n|}{|p| \times |q|}.$$

Any case in which things are not all different may be treated similarly.

Ex. 1. How many arrangements can be formed out of the letters of the word *bazaar*?

Let the required number be x .

Take one such arrangement and replace the three like letters by three unlike letters a_1, a_2, a_3 . Arrange these three unlike letters among themselves without changing the positions of the other letters. This will give $|3|$ arrangements

x arrangement treated similarly will give $x |3|$ arrangements.

But now all the six letters being different the total number of arrangements must be $|6|$.

$$\therefore x \times |3| = |6|, \text{ i.e., } x = \frac{|6|}{|3|} = 120.$$

Ex. 2. Find the number of permutations of the letters $a, \dot{a}, b, c, c, d$ taken all at a time.

Two a 's and c 's being alike the required number

$$\frac{6!}{2!2!} = 180.$$

Ex. 3. How many signals can be given by using all the ten flags, two of which are red, three blue and five yellow?

$$\begin{aligned} \text{The number of signals} &= \frac{10!}{2!3!5!} = 10 \times 9 \times 4 \times 7 \\ &= 2520. \end{aligned}$$

EXERCISE XXXIV C

1. How many arrangements can be formed out of the letters of the words :

(i) ALLAHABAD.

(ii) CALCUTTA.

(iii) CONFERENCE.

(iv) EXAMINATION (Q. 1944)

2. Find the number of permutations of the letters of the word *series*. How many of these begin and end with s and in how many are the vowels and the consonants placed alternately?

3. How many different numbers of seven digits can be formed by using the digits 1, 2, 3, 3, 4, 4, 4? How many with the digits 1, 0, 3, 3, 4, 4, 4?

4. How many numbers greater than 40,000 can be formed by using the digits 2, 4, 5, 5, 7?

5. How many different arrangements of the letters of the expression $a^2b^3c^2d$ can be made when it is written out at full length?

COMBINATIONS

8.4. Suppose there are three books and we have to select two of them for reading on a particular day. If the books are denoted by the letters a, b, c the different groups are ab, ca, bc .

Now the two books in the first group give rise to two arrangements ab and ba according as the book a is read

first or b is read first. Similar is the case with the other two groups.

Hence from three books we get three groups of two each, viz., ab, ca, bc , and the following six arrangements; $ab, ba; ca, ac, bc, cb$.

Again, if there are four books a, b, c, d and we have to select three out of them at a time, all the possible groups are

abc, abd, acd, bcd .

Each group contains three books and it gives rise to 3 or 6 different arrangements. All the 24 arrangements are as follows:—

abc	abd	acd	bcd
acb	adb	adc	bdc
bac	bad	cad	cbd
bca	bda	cda	cdb
cab	dab	dac	dbc
cba	dba	dca	dcb

Similarly if from 4 candidates, three are to be selected to form a committee, there will be 4 possible groups.

If they are further required to occupy three different offices, each group will give rise to six arrangements according to the different posts to be occupied by the candidates and there will be altogether $6 \times 4 = 24$ arrangements as shown above.

8.41. The above examples show the difference between groups and arrangements.

In forming permutations or arrangements we have to consider the order in which the things are placed, whereas in forming groups or combinations we pay attention only to the things which are contained in each selection and we have nothing to do with the order in which they are placed.

Def. Each selection or group which can be formed by taking some or all of a number of things is called a **Combination**.

Note:— If a group contains r things and if these r things are arranged among themselves, they give rise to $r!$ permutations.

tations. Hence x such groups of combinations will give us $x \times \underline{r}$ permutations.

The number of combinations of n things taken r at a time is briefly denoted by ${}^nC_r$.

8.5. To find the number of combinations of n dissimilar things taken r at a time.

Let the required number be x .

Take one of these combinations ; it contains r things which can be arranged among themselves in $\underline{r}$ ways.

One combination gives rise to $\underline{r}$ permutations.

$\therefore x$ combinations will give rise to $x \times \underline{r}$ permutations.

$\therefore x \times \underline{r} = {}^nP_r$.

$\therefore x$ or ${}^nC_r = {}^nP_r \div \underline{r}$.

Hence ${}^nC_r = \frac{{}^nP_r}{\underline{r}} = \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{r}}$.

Cor. 1. ${}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{r}} \times \frac{\underline{n-r}}{\underline{n-r}}$
 $= \frac{\underline{n}}{\underline{r} \underline{n-r}}$.

Cor. 2. Putting $r=n$ in the last result, we get

$${}^nC_n = \frac{\underline{n}}{\underline{n} \underline{0}}.$$

But ${}^nC_n$, i.e. the number of combinations of n things taken all at a time is evidently 1.

$$\therefore 1 = \frac{1}{\underline{0}}, \text{ i.e., } \underline{0} = 1.$$

Thus $\underline{0}$ (i.e., factorial zero) stands for unity.

***8. 5.** To find the value of ${}^nC_r$ without using the formula for ${}^nC_r$.
 (P. U. 1950)

Let the n things be denoted by the letters a, b, c, d , etc. and let the number of combinations of n things taken r at a time be ${}^nC_r$.

There are ${}^nC_r$ combinations and each combination contains r letters. So that if all the combinations be written down, the total number of letters so written is

$$r \times {}^nC_r \quad \dots\dots(1)$$

Now all those combinations which contain a particular letter, say a , contain also $r-1$ letters selected from the remaining $n-1$ letters, which can be done in ${}^{n-1}C_{r-1}$ ways.

Thus of all the combinations of n letters taken r at a time those that contain a particular letter a are ${}^{n-1}C_{r-1}$ in number. Therefore a is written ${}^{n-1}C_{r-1}$ times.

Similarly the letter b is also written ${}^{n-1}C_{r-1}$ times and so on for each of the n letters.

Hence total number of letters in all these combinations is also $= n \times {}^{n-1}C_{r-1}$ (2)

Hence from (1) and (2), we have $r \times {}^nC_r = n \times {}^{n-1}C_{r-1}$.

$$\text{or } {}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1}.$$

Changing n into $n-1$ and r into $r-1$, we have

$${}^{n-1}C_{r-1} = \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}$$

$$\text{Similarly } {}^{n-2}C_{r-2} = \frac{n-2}{r-2} \times {}^{n-3}C_{r-3}$$

.....

.....

$${}^{n-r+2}C_2 = \frac{n-r+2}{2} \times {}^{n-r+1}C_1.$$

• Multiplying together the vertical columns and cancelling like factors from each side and putting ${}^{n-r+1}C_1 = n-r+1$, we have

$$\begin{aligned} {}^nC_r &= \frac{n(n-1)(n-2)\dots\dots(n-r+2)(n-r+1)}{r(r-1)(r-2)\dots\dots\dots 2. 1} \\ &= \frac{n(n-1)(n-2)\dots\dots\dots(n-r+1)}{\underline{1} \underline{r}} = \frac{\underline{1} \underline{n}}{\underline{1} \underline{r} \underline{1} \underline{n-r}}. \end{aligned}$$

Ex. 1. Find n when $8 \cdot {}^nC_4 = {}^{2n}C_3$.

Here $8 \cdot \frac{n(n-1)(n-2)(n-3)}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{2n(2n-1)(2n-2)}{3 \cdot 2 \cdot 1}$

or $8n(n-1)(n-2)(n-3) = 8n(2n-1) \times 2(n-1)$

$\therefore$ since n cannot be equal to 0 or 1 (why ?), we have

$$(n-2)(n-3) = 2(2n-1)$$

or $n^2 - 9n + 8 = 0$

$\therefore n = 8$ or 1 .

Hence $n = 8$. (as n cannot be 1)

Ex. 2. In how many ways can 3 representatives to the sports committee be chosen out of 22 students ?

Clearly the required number of ways

$$= {}^{22}C_3 = \frac{22 \times 21 \times 20}{3 \times 2} = 1540.$$

Ex. 3. In how many ways can an eleven be chosen out of 13 players ?

(i) From how many of them will a particular player be excluded ?

(ii) In how many of them will he be included ?

(iii) From (i) and (ii) show that ${}^{12}C_{11} + {}^{12}C_{10} = {}^{13}C_{11}$.

Here out of 13 players we are to select eleven at a time.

$$\therefore \text{the numbers of ways} = {}^{13}C_{11} = \frac{|13|}{|11| |2|} = \frac{13 \times 12}{2} = 78.$$

(i) One player out of 13 is always excluded, i.e., selection of 11 is made out of 12.

$$\therefore \text{the required number of ways} = {}^{12}C_{11} = \frac{|12|}{|11| |1|} = 12.$$

(ii) Here one is already selected, therefore we are to select 10 more out of the remaining 12.

$$\therefore \text{the required number} = {}^{12}C_{10} = \frac{|12|}{|10| |2|} = 66.$$

(i) ${}^{12}C_{11} + {}^{12}C_{10} = 12 + 66 = 78 = {}^{13}C_{11}$. [by (1)]

Ex. 4. On how many Sundays can a party of six students be taken to the Qutab Minar from a class of 20, so that no two parties are identical ?

In how many of these parties will a particular student be included ? (D. U.)

Here we have to select 6 out of 20, therefore the total number of parties

$$= {}^{20}C_6 = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15}{6 \times 5 \times 4 \times 3 \times 2} \\ = 19 \times 17 \times 8 \times 15 = 38760.$$

All the students of any one of these parties are not the same as that of another, although some of them may be the same.

To find the number of such parties in which one particular student is included, we select 5 out of 19 and thus we have the required number $= {}^{19}C_5$

$$= \frac{19 \times 18 \times 17 \times 16 \times 15}{5 \times 4 \times 3 \times 2} = 19 \times 18 \times 17 \times 12 \\ = 11628.$$

EXERCISE XXXV

1. Evaluate :

$${}^{11}C_2, {}^{11}C_9, {}^nC_3, {}^{2n}C_n.$$

2. Find the n when ;

$$(i) {}^nP_n = 336 \text{ and } {}^nC_n = 56.$$

$$(ii) {}^nP_3 : {}^nC_5 = 1 : 5.$$

3. Show that :

$$(i) {}^{12}C_4 = \frac{1}{1 \cdot 4} \times {}^{12}P_4. \quad (ii) {}^nC_r = \frac{1}{1 \cdot r} \times {}^nP_r.$$

4. If $m = {}^nC_2$, show that ${}^mC_2 = 3 \times {}^{m+1}C_4$.

5. Find n and r from the following :

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 2 : 3 : 4.$$

6. There are 16 points in a plane (no three of them being in the same straight line). Find the number of straight lines which can be formed by joining them.

7. Find the number of triangles formed by joining the angular points of a polygon of (i) 15 sides, (ii) n sides.

8. Find the number of diagonals formed by joining the angular points of (i) an octagon, (ii) a dodecagon, (iii) a polygon of n sides.

9. There are 8 stations along a stretch of railway. Stating your argument clearly show that 56 different single third class tickets must be printed for this line.

10. How many different kinds of single second class tickets must be prepared for a new railway line with 30 stations?

11. The number of combinations of n things taken r at a time is ${}^nC_r$. In how many of these will two things occur?

12. Four persons are chosen out of ten, in how many ways this can be done and how often would any person be chosen.

(H. S. B. 1950)

13. Five persons are chosen out of ten; in how many ways can this be done and how often would any two persons be chosen?

14. A father with eight children takes three of them at a time to the Zoological gardens as often as he can without taking the same three children together more than once. How often will he go, and how often will each child go.

[See worked Ex. 4 P. 126]

15. If m be the number of combinations of n things taken 2 at a time, prove that the number of combinations of m things taken 2 together is equal to 3 times the number of combinations of $(n+1)$ things taken 4 together (See Q. 4 above)

16. Given 7 straight lines of lengths 1, 2, 3, 4, 5, 6 and 7 inches respectively, find the number of ways in which four can be chosen to form a quadrilateral. (P. U. 1924)

[Sol :—Total number of possibilities = ${}^7C_4 = 35$.

But (1, 2, 3, 6.) (1, 2, 3, 7) (1, 2, 4, 7) are the groups of straight lines in which the condition that sum of the three sides of a quadrilateral is greater than the fourth is not satisfied.

∴ Required number of ways are = $35 - 3 = 32$.]

17. A basket contains 10 mangoes (of different varieties), find how many different selections you can make of 3 mangoes so as always to include a particular mango.

18. A college council consists of the principal the vice-principal, the bursar and twelve other members. How many different sub-committees of six members can be formed including always the principal and either the vice-principal or the bursar ? (P.U.)

19. At lawn tennis six is to play against six. Find how many matches are necessary—

(i) If every player of one six is to play against every player of the other (singles) ;

(ii) if every possible pair of players of one six is to play against every possible pair of the other six (doubles).

8.6. To prove that the number of combinations of n things taken r at a time is the same as the number of combinations of n things taken $n-r$ at a time, i.e., ${}^nC_r = {}^nC_{n-r}$.

Every time we select a group of r things, we leave behind another group of $n-r$ things. Thus for every combination of $n-r$ things there corresponds a combination of r things.

$$\therefore {}^nC_r = {}^nC_{n-r}.$$

It can also be proved as follows :

$${}^nC_r = \frac{|n|}{|r| |n-r|}.$$

and
$${}^nC_{n-r} = \frac{|n|}{|n-r| |n-(n-r)|} = \frac{|n|}{|n-r| |r|}$$

Hence
$${}^nC_{n-r} = {}^nC_r.$$

Cor. Putting $r=n$, we get ${}^nC_0 = {}^nC_n = 1$.

Thus ${}^nC_0$, though meaningless in itself stands for unity.

Ex. 1. ${}^{25}C_{23} = {}^{25}C_{25-23} = {}^{25}C_2 = \frac{25 \times 24}{2} = 300.$

Ex. 2. If ${}^7C_r = {}^7C_{r+1}$ find r .

$${}^7C_r = {}^7C_{7-r} \text{ and } {}^7C_r = {}^7C_{r+1} \text{ (given)}$$

$$\therefore {}^7C_{7-r} = {}^7C_{r+1}.$$

$$\therefore 7-r = r+1, \text{ i.e., } r=3.$$

EXERCISE XXXVI-A

1. Find the values $^{200}C_{198}$, $^{78}C_{77}$, $^{78}C_0$.
2. Find the number of combinations of 100 things taken 97 at a time.
3. If $^nC_7 = ^nC_{13}$, find $^nC_{17}$.
4. If $^nC_{25} = ^nC_6$, find $^nC_{29}$ and $^{33}C_n$.
5. If $^nC_{16} = ^nC_9$, find the values of $^nC_{23}$ and $^{27}C_n$.
6. If $^{2n}C_r = ^{2n}C_{r+2}$, find r .
7. If $^nC_{r-5} = ^nC_{r+9}$ and $\frac{n}{r} = \frac{5}{2}$, find n and r .
8. If $^{n+1}C_{r+2} = 9 \times ^nC_{r+1} = 90 \times ^{n-1}C_r$, find n and r .
9. The number of combinations of $2n+1$ things taken $2n-1$ together is 105; find the number of permutations of n things taken all together.

8.61. To prove that $^nC_{r-1} + ^nC_r = ^{n+1}C_r$.

The total number of combinations of $n+1$ things taken r at a time i.e. $^{n+1}C_r$ can be divided into two groups:—

(i) Those combinations which contain a particular thing, which can be obtained by selecting $r-1$ things out of the remaining n things. Such combinations are $^nC_{r-1}$ in number.

(ii) Those which do not contain a particular thing, which can be obtained by selecting r things out of remaining n things. The number of such combinations is ^nC_r .

Hence $^nC_{r-1} + ^nC_r = ^{n+1}C_r$.

It may also be proved as follows:—

$$^nC_r = \frac{^nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$$

$$\text{and } ^nC_{r-1} = \frac{^nP_{r-1}}{(r-1)!} = \frac{\frac{n!}{(n-r+1)!}}{(r-1)!} = \frac{n!}{(r-1)!(n-r+1)!}$$

$$\therefore ^nC_r + ^nC_{r-1} = \frac{n!}{r!(n-r)!} \left[\frac{1}{r} + \frac{1}{n-r+1} \right]$$

$$= \frac{|n|}{|r-1| |n-r|} \times \frac{n+1}{r(n-r+1)}$$

$$= \frac{|n+1|}{|r| |n-r+1|} = {}^{n+1}C_r.$$

Ex. 1. Show that ${}^{12}C_4 + {}^{12}C_5 = {}^{13}C_5$.

State and prove the general theorem of which this is a particular case.

Ex. 2. From a class of 20 students select 11 such that (i) a particular student should always be included, (ii) that he should never be included.

Hence prove that ${}^{19}C_{10} + {}^{19}C_{11} = {}^{20}C_{11}$.

Ex. 3. Show that

(i) ${}^{2n+4}C_{n+2} + {}^{2n+4}C_{n+3} = {}^{2n+5}C_{n+3}$.

(ii) ${}^{n+2}C_{r+1} = {}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$.

8.7. Find the number of ways in which $p+q$ things can be divided into two groups containing p and q things.

Every time we select a group of p things, we leave behind a group of q things.

$\therefore$ the required number $= {}^{p+q}C_p = \frac{|p+q|}{|p| |q|} \dots \dots \dots (1)$

8.71. If $m+n+p$ things have to be divided into three groups of m things, n things and p things respectively, we first divide them into two groups containing $m+n$ and p things.

By (i), the number of ways in which this can be done

$$= \frac{|m+n+p|}{|m+n| |p|}$$

Again the number of ways of dividing $m+n$ things into groups containing m and n things is $\frac{|m+n|}{|m| |n|}$

$\therefore$ the number of ways of dividing $m+n+p$ things into three groups of m , n and p things,

$$= \frac{|m+n+p|}{|m+n| |p|} \times \frac{|m+n|}{|m| |n|} = \frac{|m+n+p|}{|m| |n| |p|}$$

and so on for a large number of groups.

*8.72. When $m=n=p$, the result just obtained becomes $\frac{|3p|}{(|p|)^3}$

which gives the number of ways in which $3p$ things can be divided equally into three distinct groups A, B and C.

If, however, no distinction is made between the groups, the number of ways of different sub-divisions

$$= \frac{|3p|}{|3| (|p|)^3}.$$

For example the number of ways of dealing out equally a pack of 51 cards to 3 players is $\frac{|51|}{(|17|)^3}$, but when no regard

is to be paid to the order of arrangements of the groups and the cards are to be placed in 3 heaps of 17 cards each (instead of distributing them among three persons), the number of

distribution is $\frac{|51|}{|3| (|17|)^3}$, for if x be this number, each of

these x sets of heaps of three gives rise to 3 ways of assigning one set of heaps to three persons and therefore

$$x |3| = \frac{|51|}{(|17|)^3}, \text{ i.e., } x = \frac{|51|}{|3| (|17|)^3}.$$

EXERCISE XXXVI.B

1. In how many ways can a class of 100 students be divided into two sections A and B consisting of 60 and 40 students respectively?

In how many ways can this be done when either of the section may contain 60 and the other 40 students?

2. In how many ways can 15 postal parcels be put in three bags containing 5 each, the bags being of different colours?

3. In how many ways can 9 students be equally distributed among 3 sections?

4. Prove that the number of ways of dividing up 20 unlike things into five packets of different sizes of 4 each is to the number of ways of dividing them into 4 packets of different sizes of 5 each as 625 is to 24.

5. There are 3 donkeys capable of carrying weights of 3, 4 and 5 maunds respectively ; in how many ways can 12 packets, each weighing a maund, be distributed among the donkeys ?

6. In how many ways can 12 things be equally divided among four persons. (Q. 1946)

7. Find the number of ways in which a pack of 52 cards can be dealt out to four players, giving 13 cards to each.

8. Find the number of ways in which 12 books can be divided (i) into 2 sets each containing 6 books, (ii) into 3 sets each containing 4 books.

9. On college field-day, platoon consists of 30 boys. In how many ways could it be divided into sections of 10 boys each. In how many ways could it be divided into sections of 10, each party to go off in a different direction ?

8.73. The total number of ways in which a selection can be made out of $p+q+r$ things of which p are all alike, q all alike, r all alike is

$$(p+1)(q+1)(r+1)-1.$$

There are $(p+1)$ ways of disposing of the p alike things, as we may take either 0 or 1 or 2 or 3.....or p of them, rejecting all the rest. Similarly the q alike things may be disposed of in $(q+1)$ ways and the r alike things in $(r+1)$ ways.

Hence the number of ways in which all things can be disposed of is $(p+1)(q+1)(r+1)$.

But this includes the case in which none of the things is taken ; rejecting this case, the required total number of ways is

$$(p+1)(q+1)(r+1)-1.$$

Cor. The number of ways in which a selection can be made of $p+q+r$ things of which p are alike, q all alike and the remaining r all different, is

$$(p+1)(q+1)2^r-1.$$

Ex. 1. If there are 5 different kinds of weights and three of each kind, what is the total number of different groups of weights?

The total number is

$$(3+1)(3+1)(3+1)(3+1)(3+1)-1=4^5-1=1023.$$

Ex. 2. Find the number of ways in which it is possible to make a selection by taking some or all of n different things at a time, and hence prove that ${}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n - 1$.

Here each thing may be dealt with in two ways, *viz.*, either it is rejected or accepted. But each way of dealing with a thing is associated with each way of dealing with any other.

$\therefore$ total number of ways are $2 \times 2 \dots 2 = 2^n$ but these include the particular case when none is taken.

$\therefore$ required number $= 2^n - 1$.

Again since the number of things selected may be 1 or 2 or 3.....or n , the required number of ways is also

$$= {}^nC_1 + {}^nC_2 + \dots + {}^nC_n.$$

Hence ${}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n - 1$.

Ex. 3. How many different groups of coins may be formed with a four shilling piece, a florin, a shilling and a six pence?

The number $= 2^4 - 1 = 15$.

Ex. 4. From 3 mangoes, 4 oranges and 2 apples, how many selections of fruits can be made taking (i) at least one of them, (ii) at least one of each kind? Ans. 59, 24.

Ex. 5. A question paper consists of six questions. In how many ways can a candidate attempt one or more questions? Ans 63.

Ex. 6. Prove that the total number of selections that can be made out of the letters "daddy did a deadly deed" is 1919. (B. U.)

8.8. Greatest value of ${}^nC_r$. For any particular value of n , the value of ${}^nC_r$ at first increases and then begins to decrease as r increases.

For example when (i) $n=4$ (ii) $n=5$ all the possible values of ${}^nC_r$ are

${}^4C_0,$	${}^4C_1$	${}^4C_2,$	${}^4C_3,$	${}^4C_4$
$=1$	4	6	4	1
and ${}^5C_0,$	${}^5C_1,$	${}^5C_2,$	${}^5C_3$	${}^5C_4,$
$=1,$	$5,$	$10,$	$10,$	$5,$
				${}^5C_5$
				$1.$

In the first case when $n=4$, the greatest value is 6 which occurs when $r=2=\frac{4}{2}=\frac{n}{2}$.

In the second case when $n=5$, the greatest value is 10 which occurs when $r=2=\frac{5-1}{2}=\frac{n-1}{2}$, or $r=3=\frac{5+1}{2}=\frac{n+1}{2}$.

We now discuss the general case.

***8.81.** To find the value of r for which ${}^nC_r$ i.e., the number of combinations of n different things taken r at a time, is the greatest.

We have,

$${}^nC_r = \frac{n(n-1)\dots(n-r+2)(n-r+1)}{1.2.3\dots(r-1)r},$$

$$\text{and } {}^nC_{r-1} = \frac{n(n-1)\dots(n-r+2)}{1.2\dots(r-1)}.$$

$$\therefore {}^nC_r = \frac{n-r+1}{r} \times {}^nC_{r-1}. \quad \dots(1)$$

This shows that

$${}^nC_r > {}^nC_{r-1} \text{ as long as } \frac{n-r+1}{r} > 1$$

i.e., as long as $(n-r+1) > r$ or $n+1 > 2r$

i.e., as long as $r < \frac{1}{2}(n+1)$

Since r is necessarily an integer ${}^nC_r$ is greatest when r is the integral part of $\frac{n+1}{2}$.

Two cases arise according as n is odd or even.

Case (i) when n is even, $n+1$ is odd and $\frac{n}{2}$ is the integral part of $\frac{n+1}{2}$.

$$\therefore {}^nC_r \text{ greatest when } r = \frac{n}{2}$$

$$\text{e.g., when } n=4, r = \frac{4}{2} = 2.$$

Case (ii) when n is odd, $n+1$ is even so that $\frac{1}{2}(n+1)$ is an integer, $\therefore r = \frac{n+1}{2}$.

In this case $\frac{n-r+1}{r} = \frac{2r-r}{r} = 1$, $\therefore$ from (1) ${}^nC_r = {}^nC_{r-1}$, so that $r = \frac{1}{2}(n+1)$ and $r-1 = \frac{1}{2}(n-1)$ both give the greatest value of ${}^nC_r$.

e.g. when $n=5$, $r = \frac{5+1}{2} = 3$ or $\frac{5-1}{2} = 2$.

Ex. 1. A man makes different selections of r things out of 14 things. If the number of selections is to be the greatest possible what must be the value of r ? Ans. 7

Ex. 2 A man has 12 friends whom he wishes to invite to dinner so that all the guests may not be the same at any two dinners the same number being invited each time. If he wishes to have as many dinner parties as possible, how many must he invite each time? Ans. 6

Ex. 3. A person has 8 friends, among whom he wishes to make up as many dinner parties as possible, inviting the same number each time. How many times is one particular person invited? Ans. 35

Ex. 4. If $y = {}^8C_x$, plot the positions of the point (x, y) as x takes up successive integral values. Find what value of x gives a maximum value of y . Ans. 4.

Ex. 5. Out of 9 boys how many should be taken to form a group so that the number of different groups may be the greatest? (Q. 1947) Ans. 4 or 5

8.82. To find the number of combinations of n things taken r at a time when all the n given things are not dissimilar. (Cf. Art. 8,5) and to find the number of corresponding permutations.

The method to find out the number of permutations and combinations in these cases can best be illustrated by solved examples below ;—

Example :— Find the number of (a) the combinations (b) the permutations of the letters of the word *accommodation* taken 4 at a time

(a) The letters of the word *accommodation* when written down will be as follows :

a	c	o	m	d	t	n	i
a	c	o	m				
		o					

The following cases will arise :

(i) When all the 4 letters selected are dissimilar then number of combinations $= {}^8C_4 = 70$.

(ii) When two of the letters are alike and other different then alike will be either 2 'a's or 2 'c's or 2 'o's or 2 'm's i.e. there are four possibilities and corresponding to one possibility a group for remaining two can be selected in ${}^7C_2 = 21$ ways.

$\therefore$ Total no. of combinations under this case $= 4 \times 21 = 84$.

(iii) When three are alike then only 'o's are possible and fourth can be selected in ${}^7C_1 = 7$ ways.

$\therefore$ No of combinations in this case $= 7$.

(iv) When four letters selected are like in pair's i.e. out of four groups of like things any two are taken in ${}^4C_2 = 6$ ways.

$\therefore$ Total no. of combinations $= 70 + 84 + 7 + 6 = 167$.

(b) Exactly as above we shall now form permutations in each of the different cases :—

Case (i) No. of permutations
 $= 70 \times \underline{4} = 1680$.

(as all the four are dissimilar)

(ii) No. of permutations

$$= 84 \cdot \frac{\underline{4}}{\underline{2}} = 1008$$

(as 2 are alike)

(iii) No. of permutations

$$= 7 \times \frac{\underline{4}}{\underline{3}} = 28$$

(as three are alike)

(iv) No. of permutations

$$= 6 \times \frac{4!}{2!2!} = 36.$$

(alike in two pairs)

$$\therefore \text{Total number} = 1680 + 1008 + 28 + 36 = 2752.$$

EXERCISE XXXVI-C

1. Find the number of combinations of the letters of the word **college** taken four together. (Q. 1950)

2. Find the number of combinations of the word **MATHEMATICS** taken four at a time (Q. 1948)

3. Find the number of permutations of letters of the words in Qs. 1 and 2 above when (a) all are taken together (b) when four are taken at a time.

4. Find the number of combinations and permutations of the letters of the word **independence** taken four at a time. (C. U.)

5. Find the number of combinations and permutations of the letters of the word **examination** when taken 4 at a time. (P. U.)

MISCELLANEOUS EXAMPLES

8.9. We now deal with various types of problems requiring the application of the principles of Combinations and Permutations.

8.91. Formation of a Committee.

Ex. 1. There are five professors of Mathematics and four of Physics (i) In how many ways can a committee of five be formed consisting of three professors of Mathematics and two of Physics?

(ii) In how many ways can the members take their seats in a row?

The 3 professors of Mathematics can be chosen from 5 in ${}^5C_3$ ways. and out of 4, 2 professors of Physics can be selected in ${}^4C_2$ ways.

But every group of the professors of Mathematics can be associated with every group of those of physics.

$$\begin{aligned}\text{Hence the number of committees} &= {}^5C_3 \times {}^4C_2 \\ &= 10 \times 6 = 60.\end{aligned}$$

(ii) The five members of any committee can be seated in a row in 5 ways. But there are 60 ways of forming a committee.

$$\begin{aligned}\therefore \text{the required number of ways} &= 60 \times \underline{5} \\ &= 60 \times 120 = 7200.\end{aligned}$$

Ex. 2. In how many ways can a committee of 5 be formed from 6 officials and 4 refugees representatives so as to include at least 2 refugees representatives.

A committee of 5 can consist of,

either (i) 2 refugees and 3 officials

or (ii) 3 " " 2 "

or (iii) 4 " " 1 official.

2 out of 4 refugees can be selected in ${}^4C_2$ ways and 3 out of 6 officials can be selected in ${}^6C_3$ ways.

Since each way of selecting 2 refugees can be associated with each way of selecting 3 officials, the number of ways of forming a committee as in (i).

$$= {}^4C_2 \times {}^6C_3 = \frac{4 \times 3}{2} \times \frac{6 \times 5 \times 4}{3 \times 2} = 120.$$

Similarly the number of ways in which committees can be formed as in cases (ii) and (iii) is

$${}^4C_3 \times {}^6C_2 = 4 \times \frac{6 \times 5}{2} = 60 \text{ and } {}^4C_4 \times {}^6C_1 = 1 \times 6 = 6.$$

$$\therefore \text{the required number} = 120 + 60 + 6 = 186.$$

3. Find the number of ways in which a committee of one European, 2 Americans and 3 Indians can be selected from a body consisting of 3 Europeans, 6 Americans and 9 Indians.

4. In how many ways can a committee consisting of 4 Conservatives, 3 Liberals and 2 Labourites be appointed from 10 Conservative, 5 Liberal and 3 Labourite members of a Parliament?

5. There are 8 men volunteers and 4 women volunteers. How many parties of 5 can be formed (i) including exactly one woman volunteer, (ii) including at least one woman volunteer?

8.92. Straight Lines and Triangles.

Ex. 6. There are 15 points in a plane. Find the number of (a) straight lines (b) triangles formed by joining them when (i) no three of the points are in a straight line, (ii) when 4 of the points are collinear.

(a) (i) Any two points joined together give a straight line.

$\therefore$ The number of straight lines

$$= {}^{15}C_2 = \frac{15 \times 14}{2} = 105.$$

(ii) 4 points would give ${}^4C_2 = 6$ straight lines.

But when these 4 points are collinear they form only one straight line.

Hence the required number of straight lines

$$= {}^{15}C_2 - {}^4C_2 + 1 = 105 - 6 + 1 = 100.$$

(b) Any three non-collinear points joined together form a triangle.

$\therefore$ (i) the number of triangles

$$= {}^{15}C_3 = \frac{15 \times 14 \times 13}{3 \times 2} = 5 \times 7 \times 13 = 455.$$

(ii) 4 non-collinear points form ${}^4C_3 = 4$ triangles,

but in this case 4 points being collinear give no triangle.

$\therefore$ the required number of triangles $= {}^{15}C_3 - {}^4C_3 = 455 - 4 = 451.$

7. How many triangles can be formed by joining 11 points 6 of which are in the same straight line?

8. There are p points in a plane no three of which are in the same straight line with the exception of q , which are in the same straight line; find the number of (i) straight lines; (ii) triangles which result from joining them.

8.93. Arrangement of Letters and Balls.

Ex. 9. In how many ways can the letters of the word *MAJESTIC* be arranged so that

- (i) *S* and *T* may always be together ;
- (ii) *S* and *T* may never be together ;
- (iii) the vowels may never be separated.
- (iv) the vowels as well as the consonants be never separated ;
- (v) *M* may occupy the first place and *C* the last place ?

(i) Let *S* and *T* be fastened together as one letter ; then we have to arrange seven letters *M*, *A*, *J*, *E*, (*S T*) *I*, *C*. This can be done in $|7| = 5040$ ways.

But every such arrangement gives $|2|$ separate arrangements when *S* and *T* are arranged between themselves, as *S T* and *T S*.

Hence the required number $= |7| \times |2| = 10080$.

(ii) The total number of arrangements of all the 8 letter is $|8|$.

The number of arrangements in which *S* and *T* are together is $|7| \times |2|$.

$\therefore$ the number of arrangements in which *S* and *T* are never together $= |8| - |7| \times |2| = |7| (8 - 2) = 30240$.

(iii) The three vowels may be taken together as one, then the number of letters becomes six which gives $|6|$ arrangements ; but every such arrangement gives rise $|3|$ fresh arrangements, as the three vowels in it can be arranged among themselves in $|3|$ ways. Hence the required number $= |6| \times |3| = 4320$.

(iv) The 5 consonants taken together can be arranged among themselves in $|5|$ ways and corresponding to every such arrangement there are $|3|$ arrangements of the 3 vowels.

Hence there are $|5| \times |3|$ arrangements.

But consonants and vowels as such taken separately can be arranged in $|2|$ ways.

Hence the total number of arrangements

$$= 1 \times 5 \times 1 \times 3 \times 1 \times 2 = 1440.$$

(v) The places of M and C being fixed, the remaining six letters can be arranged among themselves in ${}^6P_6 = 720$.

Ex. 10. In how many ways can the letters of the word 'FRACTION' be arranged so that (i) the vowels may occupy odd places, (ii) no two vowels may be together?

(i) In the first case we have 8 letters to arrange, 5 consonants and 3 vowels. Let us exhibit the places to be

1	2	3	4	5	6	7	8
(A)	()	(I)	()	(O)	()	()	()

The three vowels can be arranged in any three out of the four odd places marked 1, 3, 5, 7. The number of ways in which this can be done is ${}^4P_3$ or 24.

For any such arrangement of the vowels the 5 consonants can be arranged in the remaining five places in ${}^5P_5$ or 120 ways.

Since each way of arranging the vowels in the odd places can be associated with each way of arranging the consonants in the remaining places, the total number of arrangements is therefore.

$$= {}^4P_3 \times {}^5P_5 = 24 \times 120 = 2880$$

(ii) In the second case let the five consonants be arranged as shown below :—

X F X R X C X N X T X

Then in order that no two vowels may be together, the vowels can only be put in places marked as X and consonants being five in number the number of such places must be six.

Now the number of ways in which the three vowels can be arranged in any three out of these six places is ${}^6P_3$ or 120.

For each arrangement of the vowels, the 5 consonants can be arranged among themselves ${}^5P_5$ or 120 ways.

Hence the total number of arrangements

$${}^6P_3 \times {}^5P_5 = 120 \times 120 = 14400$$

Ex. 11. In how many ways can 5 white balls and 3 red balls be arranged in a row so that no two red balls are together?

Let the white balls (represented by W's) be put in a row as shown below :

$$\times W \times W \times W \times W \times W \times$$

In order that no two red balls may be together, the red balls can only be put in places marked as $\times$ and the white balls being five, the number of such places is six.

Hence in order to place the 3 red balls, we must choose any 3 out of these six places ; which can be done in ${}^6C_3$ or 20 ways.

Note Here the balls being alike the question of arrangement of the white and red balls among themselves does not arise.

12. In how many ways can 8 answer-books be arranged so that the best book and the worst book are always together ?

13. Find the number of ways in which 3 books on Mathematics, 4 on Physics and 2 on Chemistry can be arranged on a shelf so that books on the same subject are together.

14. In how many ways can the letters of the word 'fraction' be arranged so that

- (i) f and r may always be together ;
- (ii) f and r may never be together ;
- (iii) the vowels may never be separated ;
- (iv) the vowels as well as the consonants may be never separated ;
- (v) f may occupy the first place and n the last place ;
- (vi) f and n may occupy the end places ?

15. How many arrangements can be made from the letters of the word 'PENCIL'.

- (i) beginning with N and ending with C ;
- (ii) N and C occupying end place ;
- (iii) having N, C always together ;
- (iv) having N, C never together ;
- (v) so that N, and C occupy middle places ;
- (vi) N and C as in (v), letters P, E and I, L coming always before or after N, C ?

16. In how many ways can the letters of the word *terror* be arranged so that

(i) the two vowels are together ;

(ii) the relative position of the vowels and consonants is not altered ?

17. How many numbers can be formed with the digits 9, 6, 5, 5, 2, 6, 1, so that the even digits may *always* occupy even places ?

18. In how many ways can the letters of the word "integral" be arranged so that no two vowels may be together ?

19. Find how many different words can be formed with five given letters, of which three are consonants and two are vowels, no two consonants being juxtaposed in any word.

20. In how many ways can 12 cricket balls and 7 tennis balls be arranged in a row so that no two tennis balls are together ?

8.94. Combination and permutations including particular things.

Ex. 21. In how many of the (i) combinations (ii) permutations of n things taken r at a time will 4 given things occur ?

(i) Since 4 particular things are to be included in each group, we have to select only $r-4$ out of the remaining $n-4$.

Hence the required number $= {}^{n-4}C_{r-4}$.

(ii) The r things in each of the ${}^{n-4}C_{r-4}$ groups can be arranged among themselves in $|r|$ ways

$\therefore$ the required number of permutations $= {}^{n-4}C_{r-4} \times |r|$

which can also be written as

$$\frac{|n-4|}{|n-r| |r-4|} \times |r| = \frac{|n-4|}{|n-r|} \times \frac{|r|}{|r-4|} \text{ or } {}^{n-4}P_{r-4} \times {}^rP_4.$$

$$[\therefore {}^nP_r = \frac{|n|}{|n-r|}]$$

Ex. 22. Prove that ${}^n P_r = {}^{n-1} P_r + r \times {}^{n-1} P_{r-1}$. (P. U. 1933)

${}^n P_r$, i.e., the number of permutations of n different things taken r at a time can be taken in two groups :—

(i) Those permutations which do not contain a particular thing :—such permutations can be obtained by arranging the remaining $n-1$ things in r places and therefore the number is ${}^{n-1} P_r$.

(ii) Those permutations in which one particular thing always occurs ;—The number of such permutations is

$${}^{n-1} C_{r-1} \times r = {}^{n-1} C_{r-1} \times (r-1) \times r = {}^{n-1} P_{r-1} \times r$$

$$\text{Hence } {}^n P_r = {}^{n-1} P_r + r \times {}^{n-1} P_{r-1}.$$

Ex. 23. If ${}^n P_r$ denotes the number of permutations of n things taken r at a time, prove from first principles that

$${}^n P_r = n \times {}^{n-1} P_{r-1}.$$

Hence find the value of ${}^n P_r$ as a product. (P. U. 1944)

Let one of the n things occupy a certain fixed place, then we have to arrange the remaining $(n-1)$ things in $r-1$ places, i.e., we have to find the number of permutations of $(n-1)$ things taking $(r-1)$ of them at a time. These permutations must be ${}^{n-1} P_{r-1}$ in number.

Now if we repeat the same thing with any other particular thing, we again get the same number of permutations.

Hence the total number of permutations $= n \times {}^{n-1} P_{r-1}$ but this is also $= {}^n P_r$.

$$\text{Hence } {}^n P_r = n \times {}^{n-1} P_{r-1}.$$

Decreasing both n and r by 1 repeatedly, we get

$${}^{n-1} P_{r-1} = (n-1) \times {}^{n-2} P_{r-2}$$

$${}^{n-2} P_{r-2} = (n-2) \times {}^{n-3} P_{r-3}$$

$$* \quad * \quad * \quad * \quad *$$

$${}^{n-r+3} P_3 = (n-r+3) \times {}^{n-r+2} P_2$$

$${}^{n-r+2} P_2 = (n-r+2) \times {}^{n-r+1} P_1.$$

Multiplying all and cancelling common terms on the two sides we have

$$\begin{aligned} {}^n P_r &= n(n-1)(n-2) \dots (n-r+2) {}^{n-r+1} P_1 \\ &= n(n-1)(n-2) \dots (n-r+2)(n-r+1). \end{aligned}$$

Ex. 24. A man possesses 10 coins out of which he has got to take 5 coins. Find in how many of the selections (i) one (ii) both of two given coins will occur.

Ex. 25. Show that the number of permutations of n things taken r together in which p particular things always occur is

$${}^{n-p}P_{r-p} \times {}^pP_p.$$

8.95. Permutations in a ring.

Ex. 26. In how many ways can 5 persons form a ring?

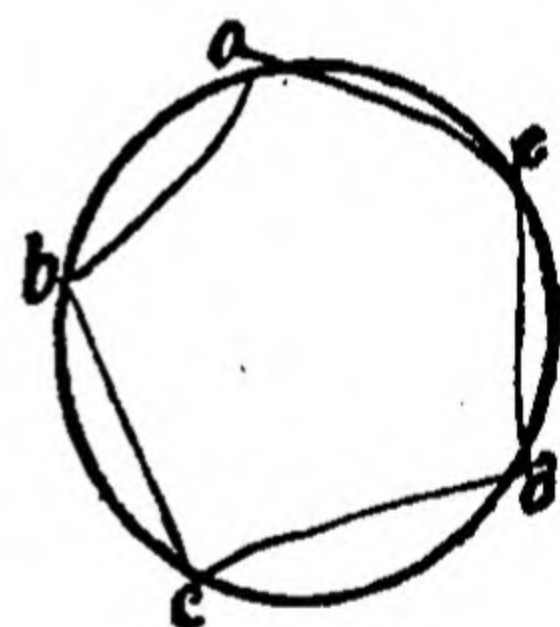
Let the 5 persons be denoted by the letters a, b, c, d , and e , and let one of the ways in which they can form a ring be as shown in the figure.

Starting with different letters and reading them in the counter-clockwise direction the various arrangements of the letters thus obtained are

$abcde, bcdea, cdeab, deabc, eabcd.$

These are *different linear arrangements*, but have no essential difference when regarded as *circular arrangements*.

This shows that a single circular arrangement of the 5 letters gives rise to 5 different linear arrangements.



Hence if the required number of circular arrangements of the 5 persons be x , the total number of the linear arrangements of the same persons will be $5x$.

But we know that the total number of linear arrangements of 5 persons is $|5|$.

Hence $5x = |5|$.

$$\therefore x = \frac{1}{5} |5| = |4|.$$

Thus the number of ways in which *five* persons form a ring = $|4| = |5-1|$.

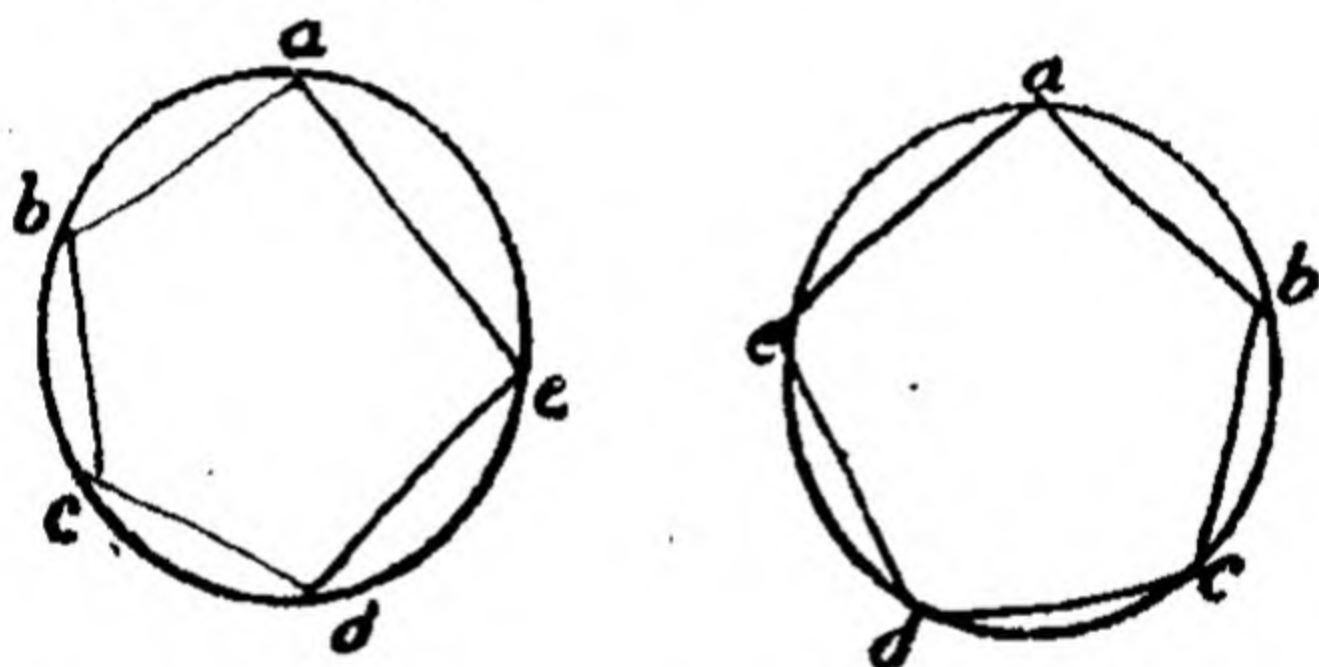
Similarly the number of ways in which n persons can form a ring = $|n-1|$.

The same results can easily be obtained as follows :—

Since in each circular arrangement we have to consider the position of one person relatively to others, we can get the desired result by fixing some person in one position and then by arranging the remaining $n-1$ persons among themselves in $(n-1)!$ ways.

Ex. 27. In how many ways can 5 persons be seated at a round table so that all shall not have the same neighbours in any two arrangements, (Q. 1951)

The total number of ways in which 5 persons can take their seats at a round table, (one of them occupying a fixed position) $= (5-1)! = 4! = 24$.



Two of these 24 arrangements are shown in the above figures.

It is evident that in both these arrangements all the 5 persons have got the same neighbours only with this difference that the left-hand neighbour in the one becomes the right-hand neighbour in the second.

Since all persons are not to have the same neighbours such a pair of arrangements must be counted as one.

$\therefore$ the required number $= \frac{1}{2} \times 24 = 12$.

Ex. 28. Find the number of ways in which 3 men and 3 women can be seated at a round table so that no two women may be together.

Let the 3 men occupy seats with alternate places left vacant. This can be done in $(3-1)! = 2$ ways.

The 3 vacant places can now be filled by women in 3 different ways $\therefore$ the required number = $|2 \times |3 = 2 \times 6 = 12$.

29. In how many ways can 8 persons be seated, at a round table?

30. In how many ways can four persons be seated on a council table so that all shall not have the same neighbours in any two arrangements? (Q. 1945)

31. (a) In how many ways can ten children sit in a merry-go-round relative to one another?

(b) In how many of these arrangements shall all children have different children in front of them, the merry-go-round being revolved in either direction?

32. Given ten heads of different colours, how many different necklaces may be formed by stringing them all together?

33. Find the number of ways in which 12 persons can form a ring so that 3 may always be together?

34. In how many ways can 8 gentlemen and 8 ladies sit down at a round table so that no two ladies may be together?

35. A gentleman invites a party of $m+n$ friends to dinner and places m at one round table and n at another. Show that the number of ways in which he can arrange them among themselves is

$$\frac{|m+n|}{m.n}$$

36. A gentleman invites a party of 13 guests to a dinner and places 8 of them at one table and the remaining 5 at another, the tables being round. Find the number of ways in which he can arrange the guests.

37. In how many ways can 5 rupees and 6 pice be arranged (i) in a row, (ii) in a ring? (P. U. 1937)

EXERCISE XXXVII

1. There are seven animals, one of them being a cat and another a rat. In how many ways can they go in a line so that the cat and the rat are never together?

2. There are six clerks in a college, three senior and three junior. In how many ways can the office work be assigned to them in three months' vacation if it is required that one senior and one junior clerk must be in the office each month?

[Hint. 3 senior clerks can be assigned three months in 3 ways and so also the juniors].

3. Twenty-four persons are to travel by a motor-bus which can accommodate 15 inside and 9 outside. If 6 object to travel outside, and four are unwilling to travel inside, find the number of different ways in which the parties can travel.

[Hint. Give the $6+4=10$ unwilling persons their full choice, out of the remaining $24-10=14$ select either 9 for inside or 5 for outside.]

4. 10 M.A.'s and 7 M.Sc.'s apply for admission to the B.T. class out of which only 5 are to be taken. In how many ways can the selection be made if at least one M.Sc. must be taken?

5. Two rings which are exactly alike have to be placed on ten pegs of different colours each of which is capable of taking both rings. Find the number of ways in which this can be done.

6. A girl has three different rings. Find in how many ways she can arrange them on her eight fingers with

(i) not more than one ring on any finger ;

(ii) not more than two rings on any finger. (D.U)

7. How many parcels, each containing not more than 6, can be made with 8 books to choose from?

8. At an election where every voter may vote for any number of candidates not greater than the number to be chosen, there are 6 candidates and 3 members to be chosen; in how many ways can a man vote?

9. There are 4 candidates for two vacancies. There are 3 electors, each of whom can vote for two candidates or just for one; in how many ways can their votes be given?

10. Show that the number of all possible selections of one or more questions from 9 given questions is $2^9 - 1$.

[Hint. Either a question is selected or rejected, i.e., each question is dealt with in two ways. Also each way of dealing with a question is associated with each way of dealing with any other. Therefore the number of ways is

$$= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9$$

But this includes the case when all are rejected.

$\therefore$ the required number $= 2^9 - 1$.]

11. Show that the number of all possible selections of one or more questions from eight given questions, each question having an alternative, is $3^8 - 1$.

12. A telegraph has 5 arms and each arm is capable of 4 distinct positions including the position of rest; what is the total number of signals that can be made?

13. An examination paper is divided into two groups consisting of 3 and 4 questions. Candidates are required to answer at least one question from each group. In how many ways can a candidate select the questions?

14. How many words can be formed out of the letters of the word *article*, so that the vowels may occupy the even places. (P U.)

15. In how many ways can the letters of the word (i) *integral* (ii) *chairman* be arranged so that the vowels may occupy odd places?

16. Find the number of ways in which the letters of the word (i) *fashion* (ii) *mansion* can be arranged such that the vowels may occupy odd positions.

17. Find the number of words that can be formed with two different consonants and one vowel out of 7 different consonants and 3 different vowels, the vowel to be between the two consonants

18. Show that the number of ways of seating $2n$ persons at two round tables, n at each, is $(2n-1) \times n!$.

19. Show that the product of r consecutive integers is divisible by $r!$.

[Hint This product is ${}^nP_r = \frac{n!}{(n-r)!}$.]

20. Show that

$$\frac{n(n+1)(n+2)\dots(n+r-1)}{r!} \text{ is an integer.}$$

[Hint. This $= {}^{n+r-1}C_r$.]

21. There are m men and n monkeys; n being greater than m ; find the number of ways in which each man becomes the owner of one monkey.

If a man may have any number of monkeys, in how many ways may every monkey have a master?

22. A rectangle has n lines ruled across it parallel to one pair of sides and n lines parallel to the other sides. Prove that the number of rectangles formed is $\frac{1}{2}(n+1)^2(n+2)^2$.

23. On three straight lines are taken a , b and c points respectively, (*i. e.*, a points on one line, b on another, and c on the third); no one of the points is a point of the intersection of the lines and no three of the points, one on each line are collinear. Prove that the number of triangles which can be made by taking three of the points is:—

$$\frac{1}{2}(b+c)(c+a)(a+b) - (ab+bc+ca).$$

REVISION QUESTIONS III

1. Find the number of permutations of n things taken r at a time. How many of these will (i) include two things, (ii) exclude two things?

2. How many arrangements can be made out of the letters of the word "draught" the vowels never being separated?

3. Prove that the number of permutations of n letters taken r at a time which begin with some particular letter is ${}^{n-1}P_{r-1}$.

(a) Hence show that ${}^nP_r = n \times {}^{n-1}P_{r-1}$.

(b) Deduce the formula ${}^nP_r = n(n-1)(n-2)\dots(n-r+1)$.

4. Show that 154 numbers less than 1000 and divisible by 5 can be formed with the ten digits, each digit not occurring more than once in each number.

5. Find the number of permutations of the letters in the word INDIA.

6. There are p copies of each of the n different books ; show that the number of the different orders in which they can be arranged on a shelf is $\frac{n!}{p!} p^n$.

7. There are $4n$ things of which n are alike and all the rest different, show that the number of permutations of the $4n$ things taken $2n$ together, each permutation containing the n alike things is $\frac{(4n)!}{(2n)! n!}$.

8. Show that the number of ways in which it is possible to make a selection by taking some or all out of $p+q+r$ things, where of p are alike of the one kind, q alike of second kind and r alike of a third kind, is

$$(p+1)(q+1)(r+1)-1$$

9. There are n different books and p copies of each, show that the number of ways in which a selection can be made from them is $(p+1)^n-1$.

10. Show that the total number combinations out of $(p+q+r)$ things of which p are alike and q alike and r all different, is $(p+1)(q+1)2^r-1$.

11. Show that (i) the number of ways in which mn things can be divided into m sets of n things is

$$\frac{(mn)!}{m! n^m}$$

(ii) the number of ways in which $2n$ things can be divided into groups of n pairs is $\frac{(2n)!}{2^n n!}$.

12. A person has 15 acquaintances of whom 5 are relatives. In how many ways may he invite 13 guests from among them so that 3 of these may be relatives ?

13. In how many ways can 16 rupees be divided among 4 persons so that no person shall receive less than 3 rupees ?

14. In how many ways can 16 departments be allotted to 4 ministers so that no minister shall have less than 3 departments under him ? (Departments to be regarded as alike).
(P. U. 1939)

15. In how many ways can 10 different toys be divided among 5 children so that each child may receive two toys ?

16. If there be two kinds of balls, red and green, and at least five of each kind, how many different ways are there of filling five different boxes with one ball each ?

17. How many different numbers greater than 10,000 can be made from the figures of the number 55301 ?

18. Show that ${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1}$ and hence find the value of ${}^nC_r$.
[See Art. 8.5]

19. Show that

$$(i) \quad |2n| = \{ 2.3.5.7.....(2n-1) \} 2 \times |n|.$$

$$(ii) \quad |4n| = \{ 1.3.5....(4n-1) \} \times 2^{2n} \times |2n|.$$

20. Find the sum of all numbers that can be formed by taking all digits at a time using the digits 1, 2, 3, 4, 5, no digit being repeated in any number.

[Hint. It is clear that in 4 i.e., 24 cases the place of any one of the digits, say 1, is fixed, (i.e., the digit 1 occupies the unit's place in 24 cases, the ten's place in 24 cases and so on). Therefore the sum arising from the digit 1 alone is

$$= 24[1 + 10 + 100 + 1000 + 10000] \\ = 266664.$$

$$\text{Similarly sum from 2 alone} = 2 \times 266664$$

$$\text{,, ,, ,, 3 ,,} = 3 \times 266664$$

$$\text{,, ,, ,, 4 ,,} = 4 \times 266664$$

$$\text{,, ,, ,, 5 ,,} = 2 \times 266664 \text{ etc. Now add.}$$

21. Find the sum of all numbers greater than 1000 formed by using the digits 1, 4, 6, 9 no digit being repeated in any number.

22. Find the number of ways of carrying all of 4 flags on 3 masts, where all the masts need not be used.

23. Out of a party of 4 first net and 3 second net players one game at lawn tennis is to be arranged, each side consisting of one first net player and one second net player. In how many ways can this be done ?

24. A tennis club consists of 6 single ladies, 6 single gentlemen and a married couple. In how many ways can a mixed double (*i.e.*, one lady and one gentleman on each side) be selected so that the married couple are not both included ?

25. Prove from first principles that

$${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1} \quad [\text{See Art. 8.61}]$$

Apply it to show that

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n. \quad (\text{P. U. 1932})$$

[Sol. We have

$${}^nC_0 = {}^{n-1}C_0 \text{ (each being unity)}$$

$${}^nC_1 = {}^{n-1}C_1 + {}^{n-1}C_0 \text{ (by part first of the question)}$$

$${}^nC_2 = {}^{n-1}C_2 + {}^{n-1}C_1$$

.....

$${}^nC_{n-1} = {}^{n-1}C_{n-1} + {}^{n-1}C_{n-2}$$

$${}^nC_n = {}^{n-1}C_{n-1} \text{ (each being unity).}$$

$$\begin{aligned} \text{Adding } {}^nC_0 + {}^nC_1 + \dots + {}^nC_n &= 2({}^{n-1}C_0 + {}^{n-1}C_1 + \dots + {}^{n-1}C_{n-1}) \\ &= 2^2({}^{n-2}C_0 + {}^{n-2}C_1 + \dots + {}^{n-2}C_{n-2}). \end{aligned}$$

etc.....etc.....

$$= 2^{n-1}({}^1C_0 + {}^1C_1)$$

$$= 2^{n-1} \times 2$$

$$= 2^n.$$

CHAPTER IX

BINOMIAL THEOREM

Positive integral Index

9.1. An algebraical expression consisting of two terms connected by the signs $+$ or $-$ such as $x+a$, $3x^2-5y$, is called a **Binomial**.

The product of a binomial by itself any number of times, or raising a binomial to any power or the root of a binomial is very often required. It can easily be done by the help of a general formula called the **Binomial Theorem**.

We now prove a special case of this important theorem by the method of Induction already explained.

9.2. To prove that when n is a positive integer

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + \dots + {}^nC_r x^{n-r}a^r + \dots + {}^nC_n a^n.$$

By actual multiplication we have

$$(x+a)^2 = x^2 + 2ax + a^2 = {}^2C_0 x^2 + {}^2C_1 xa + {}^2C_2 a^2$$

$$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3 \\ = {}^3C_0 x^3 + {}^3C_1 x^2a + {}^3C_2 xa^2 + {}^3C_3 a^3.$$

Thus we find that the theorem is true for $n=2$ and $n=3$.

Let us now assume that it is true for a particular integral power m ; i.e., let

$$(x+a)^m = {}^mC_0 x^m + {}^mC_1 x^{m-1}a + {}^mC_2 x^{m-2}a^2 + \dots \\ \dots + {}^mC_{r-1} x^{m-r+1}a^{r-1} + {}^mC_r x^{m-r}a^r + \dots + {}^mC_m a^m. \quad (1)$$

Multiplying both sides by $(x+a)$ and collecting like terms, we get

$$(x+a)^{m+1} = {}^mC_0 x^{m+1} + ({}^mC_0 + {}^mC_1)x^m a + ({}^mC_1 + {}^mC_2)x^{m-1}a^2 + \dots \\ + ({}^mC_{r-1} + {}^mC_r)x^{m-r+1}a^r + \dots + {}^mC_m a^{m+1}.$$

But ${}^mC_0 = 1 = {}^{m+1}C_0$, ${}^mC_m = 1 = {}^{m+1}C_{m+1}$
and ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$;

$$\therefore (x+a)^{m+1} = {}^{m+1}C_0 x^{m+1} + {}^{m+1}C_1 x^m a + \dots + {}^{m+1}C_r x^{m+1-r} a^r + \dots + {}^{m+1}C_{m+1} a^{m+1}. \quad (2)$$

We notice that the successive terms in (2) are of the same form as in (1) except that m is changed into $m+1$.

Hence if the theorem is true for the power m it is also true for the power $m+1$.

But the theorem is true for $n=2$.

Hence it is true for $n=2+1=3$ and being true for 3, it must be true for the next integer 4 and so on.

Hence the theorem is true for all positive integral values of n .

9.2. The following facts about the Binomial expansion may be noted :

(i) The number of terms is one more than the power of the binomial.

(ii) The power of x goes on decreasing by one and that of a goes on increasing by one, the highest power of x and the sum of the powers of x and a being always equal to the power of the Binomial.

(iii) The co-efficient of any term is the number of combinations formed from a number of things equal to the power of the binomial, the things taken being equal to the power of a (or one less than the order of term).

9.3. The $(r+1)$ th term in the expansion of the Binomial Theorem is called the **General Term**.

It is evident from the expansion of $(x+a)^n$ given in Art. 9.2 that

$$T_{r+1} = {}^nC_r x^{n-r} a^r.$$

9.4. Replacing x by 1 and a by x , we get

$$\begin{aligned}(1+x)^n &= 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n \\ &= 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{3} x^3 \\ &\quad + \dots + x^n,\end{aligned}$$

and writing $-x$ for x in this we get

$$(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - \dots + (-1)^n x^n.$$

9.5. **Examples.** *is better than principle.*

Ex. 1. Expand $(2x+3y)^5$.

$$(2x+3y)^5 = (2x)^5 + 5(2x)^4(3y) + \frac{5.4}{1.2} (2x)^3(3y)^2 +$$

$$\frac{5.4.3}{1.2.3} (2x)^2(3y)^3 + \frac{5.4.3.2}{1.2.3.4} (2x)(3y)^4 + \frac{5.4.3.2.1}{1.2.3.4.5} (3y)^5.$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.$$

Ex. 2. Write down the 10th term in the expansion of $(2x-y)^{11}$.

By Art 9.3, we have $T_{10} = {}^{11}C_9 (2x)^{11-9} (-y)^9 = {}^{11}C_2 4x^2y^9 (-1)^9$

$$= -\frac{11 \times 10}{1 \times 2} 4x^2y^9 = -220x^2y^9.$$

Ex. 3. Find the middle term of $\left\{ x - \frac{1}{x} \right\}^{10}$.

The total number of terms in the expansion being eleven, the 6th term shall be the middle term.

$$T_6 = {}^{10}C_5 x^{10-6} \left\{ -\frac{1}{x} \right\}^6 = \frac{10.9.8.7.6}{5.4.3.2.1} x^5 (-1)^6 \frac{1}{x^6} = -252.$$

Ex. 4. Find the two middle terms of $\left\{ 2a - \frac{a^2}{4} \right\}^9$.

The total number of terms being 10 ; 5th. and 6th will be the two middle terms.

$$T_5 = {}^9C_4 (2a)^{9-4} \left(-\frac{a^2}{4}\right)^4 = \frac{9.8.7.6}{4.3.2.1} 2^5 a^5 \cdot \frac{a^8}{4.4.4.4} = \frac{63}{4} a^{13}.$$

$$T_6 = {}^9C_5 (2a)^{9-5} \left(-\frac{a^2}{4}\right)^5 \\ = \frac{9.8.7.6.5}{5.4.3.2.1} \cdot \frac{2^4 a^4 (-1)^5 a^{10}}{4^5} = -\frac{63}{32} a^{14}.$$

EXERCISE XXXVIII

Expand the following :

1. (i) $(1+x)^4$, (ii) $(1-x)^5$.

2. (a) (i) $\left(2x - \frac{y}{3}\right)^6$, (ii) $\left(\frac{x}{y} + \frac{y}{x}\right)^6$.

(b) Prove that $\left(x + \frac{1}{x}\right)^6 - \left(x - \frac{1}{x}\right)^6 = 12\left(x^4 + \frac{1}{x^4} + \frac{10}{3}\right)$

3. Find the (i) 11th term of $\left(\frac{3x}{2} + \frac{2a}{3}\right)^{17}$.

(ii) 19th term of $\left[2a^{\frac{1}{2}} - b^{\frac{1}{3}}\right]^{20}$.

4. (a) Find the term independent of x in $\left(3x^2 - \frac{a}{3x^3}\right)$.

(b) Find the term independent of x in $\left(x - \frac{1}{x^2}\right)^{20}$.

5. Find the middle term in the expansion of

(i) $(1+x)^{110}$. (ii) $\left(\frac{4a}{3} - \frac{3x}{2}\right)^8$.

6. Find the two middle terms of :—

(i) $\left[3x - \frac{x^3}{6}\right]^9$ (ii) $\left(x - \frac{3}{x}\right)^{11}$ (iii) $(a+x)^{2n+1}$.

7. If $\left(x + \frac{p}{y}\right)^n = x^6 - 3 \frac{x^5}{y} + \frac{15x^4}{4y^2} + \dots$;

find the value of p and n .

8. In the expansion of $(\sqrt{3} + \sqrt{2})^6$ find which terms are rational ?

9. Find the ratio of the middle term to the one before it, in the expansion of $(1 - 2x)^{20}$.

10. If in the expansion of $(1 + x)^n$ the fifth term be four times the fourth term and the fourth term be six times the third term, find the values of n and x .
(Delhi H. S.)

9.6. Examples :—

Ex 1. Find the coefficient of x^n in the expansion of $(1 + x)^{2n}$, n being a positive integer.

Here $T_{r+1} = {}^{2n}C_r x^{2n-r}$.

But since we have to find the coefficient x^n , we put

$$2n - r = n$$

$$\therefore r = n.$$

$$\therefore \text{the required coefficient} = {}^{2n}C_n = \frac{{}^{2n}}{(n)}.$$

Ex. 2. Write down the co-efficient of $\frac{1}{y^2}$ in the expansion of $\left(y + \frac{a^2}{y^2}\right)^{10}$.

$$T_{r+1} = {}^{10}C_r y^{10-r} \left(\frac{a^2}{y^2}\right)^r = {}^{10}C_r y^{10-r} a^{2r} y^{-2r} = {}^{10}C_r a^{2r} y^{10-3r}.$$

If this term contains $\frac{1}{y^2}$ or y^{-2} we must have

$$10 - 3r = -2, \text{ i.e., } r = 4.$$

Putting $r = 4$: we get

$$T_5 = {}^{10}C_4 a^8 y^{10-12} = \frac{10.9.8.7}{4.3.2.1} a^8 \cdot \frac{1}{y^2} = \frac{210a^8}{y^2}.$$

$$\therefore \text{the co-efficient of } \frac{1}{y^2} = 210a^8.$$

Ex. 3. Find the term independent of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$.

Let $(r+1)$ th term be independent of x .

$$\begin{aligned}\text{Then } T_{r+1} &= {}^{12}C_r (2x^2)^{12-r} \left(-\frac{1}{x}\right)^r \\ &= {}^{12}C_r 2^{12-r} x^{24-2r} (-1)^r x^{-r} \\ &= {}^{12}C_r (-1)^r 2^{12-r} x^{24-3r}.\end{aligned}$$

This will be independent of x when

$$\begin{aligned}24-3r &= 0, \text{ i.e., } r=8. \text{ Putting } r=8, \text{ we get} \\ T_9 &= {}^{12}C_8 (-1)^8 2^4 x^0 = 16 \times {}^{12}C_4 \\ &= 16 \times \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 16 \times 495 = 7920.\end{aligned}$$

Ex. 4. Find the Binomial expansion whose first three terms are 32, 240, 720.

Let the binomial expansion be $(a+x)^n$.

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}x^3 + \dots$$

Therefore comparing the first three terms, we have

$$a^n = 32 \quad (i)$$

$$na^{n-1}x = 240 \quad (ii)$$

$$\frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 = 720 \quad (iii)$$

Thus we have three equations in 3 unknowns a , x and n .
Multiplying (i) and (iii) and dividing by the square of (ii) we have

$$\frac{n(n-1)}{1 \cdot 2} \times \frac{a^{2n-2}x^2}{n^2 a^{2n-2}x^2} = \frac{720 \times 32}{240 \times 240}$$

$$\text{or } \frac{n-1}{2n} = \frac{2}{5} \text{ or } 5n-5=4n, \text{ or } n=5.$$

Putting $n=5$ in (i), we get $a^5=32=2^5$; $\therefore a=2$.

Putting $a=2$, $n=5$ in (ii), we get

$$5 \times 2^4 \times x = 240 \text{ so that } x=3.$$

Thus the required binomial expansion is $(2+3)^5$.

EXERCISE XXXIX

1. Do (i) x^{16} and (ii) x^{18} occur in the expansion of $(a^4 - bx^3)^{10}$? If so, find their co-efficients.

2. Find the co-efficient of x^{16} in $\left(2x + \frac{1}{x^3}\right)^{36}$.

3. Find the co-efficient of x^{17} in the expansion of $(x^2 - 2x)^{12}$.

4. Show that the co-efficient of y^{22} and y^{-6} in $\left[y^4 - \frac{1}{y^3}\right]^{16}$ are equal.

5. (i) Find the co-efficient of x^4 in the expansion of $(1+x-2x^2)^6$.

(ii) and of x^{15} in $(x-x^2)^{10}$. (H. S. B. 1947)

6. Find the two middle terms of the expansion of $\left(x - \frac{1}{x}\right)^{11}$. (H. S. B. 1948)

7. Write down 5th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^8$ and show that it does not contain x . (H. S. B. 1948)

8. Find the co-efficient of x^7 in the expansion of $(1-x)^4(1+x)^9$.

9. Find the term independent of x in the following :

$$(i) \left[3x^2 - \frac{b}{3x^3}\right]^{10} \quad (ii) \left(2x^2 + \frac{1}{x}\right)^{12}$$

$$(iii) \left(x^2 - \frac{2}{x}\right)^9 \quad (iv) \left(x + \frac{1}{x}\right)^{10}$$

$$(v) \left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9 \quad (Q. 1950) (H. S. B. 1948)$$

10. Find the term independent of x in the expansion of $\left(x^3 - \frac{1}{x^2}\right)^{10}$. Find also the sum of all the terms when $x=1$.

(D. U. 1943)

11. Show that the co-efficient of x^n in the expansion of $(1+x)^{2n}$ is double the co-efficient of x^n in the expansion of $(1+x)^{2n-1}$.

12. Find r when the co-efficient of the $(r-1)$ th and $(2r+3)$ th terms of $(1+x)^{15}$ are equal.

13. If a_r denote the co-efficient of x^r in $(1-x)^{2n-1}$, prove that $a_{r-1} + a_{2n-r} = 0$.

14. Find the co-efficient of x^9 in the expansion of $(1-x^3)^{10}(1+x)^6$.

15. Find the value of the term independent of x in the expansion of $\left(x^3 - \frac{2}{x^3}\right)^5 (2+x)^4$.

16. The expansion of $(1+ax)^n$ is $C_0 + C_1x + C_2x^2 + \dots + C_rx^r + \dots + C_nx^n$, n being greater than 3 and a is not zero. If $2C_2 = 3C_3$ and $C_1C_3 = 7C_4$ find a and n .

17. The first three terms in the expansion of $(1+ax)^n$ by the binomial theorem are 1, $8x$, $30x^2$. Find the values of a and n and show that the co-efficient of x^5 in the expansion is $136\frac{1}{2}$.

18. Find the binomial expansion of which four consecutive terms are 576, 2160, 4320, 4860. (P. U. 1931)

19. Employ the Binomial Theorem to find $(999)^3$. (H S. B. 1946)

9.7. Properties of the Binomial Co-efficients

The co-efficients of various terms in the expansion of $(x+a)^n$ or $(1+x)^n$ are

$${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \dots$$

$$\text{or } 1, n, \frac{n(n-1)}{1 \cdot 2}, \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, \dots$$

These are called the Binomial Co-efficients. We shall now consider the properties of these co-efficients.

(1) *The co-efficients of terms equidistant from the beginning and end are equal.*

The co-efficient of the $(r+1)$ th term from the beginning is ${}^nC_r$.

There are altogether $(n+1)$ terms and the $(r+1)$ th term from the end has $(n+1) - (r+1)$ or $n-r$ terms before it.

Hence the $(r+1)$ th term from the end is $(n-r+1)$ th term from the beginning and its co-efficient is ${}^nC_{n-r}$.

But ${}^nC_r = {}^nC_{n-r}$.

$\therefore$ the co-efficients of the $(r+1)$ th terms from the beginning and end are equal.

(2) The sum of the Binomial Co-efficients $= 2^n$.

Put $x=1$ in the expansion

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n.$$

We get

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n.$$

$=$ sum of all the co-efficients.

(3) The sum of the co-efficients of the odd terms $=$ the sum of the co-efficients of the even terms $= 2^{n-1}$.

Put $x=-1$ in the expansion

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n.$$

We get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots$$

$$\text{i.e., } {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \dots$$

$$= \frac{1}{2} \times 2^n = 2^{n-1}.$$

Ex. 1. Show that

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^{n-1}.$$

Interpret this result.

Ex. 2. Show that in the expansion of $(1-x)^n$ the sum of the co-efficients is equal to zero.

Ex. 3. Show that the sum of the even co-efficients of the terms in the expansion of $(1+x)^{2n}$ is 2^{2n-1} .

Ex. 4. Show that

$$(i) \quad 1 + n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \dots + 1 = 2^n.$$

$$(ii) \quad 1 + (n-1) + \frac{(n-1)(n-2)}{1 \cdot 2} + \dots + 1 = 2^{n-1}.$$

9.8. Examples

Ex. 1. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$; * find the values of

$$(i) \quad C_1 + 2C_2 + 3C_3 + \dots + n.C_n;$$

(P. U. 1950)

$$(ii) \quad \frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \dots + \frac{n.C_n}{C_{n-1}}$$

$$(i) \quad C_1 + 2C_2 + 3C_3 + \dots + n.C_n$$

$$= n + 2 \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \dots + n.$$

* $C_0, C_1, \dots, C_n$ stand for ${}^nC_0, {}^nC_1, \dots, {}^nC_n$.

$$= n \left[1 + (n-1) + \frac{(n-1)(n-2)}{2} + \dots + 1 \right]$$

$$= n(1+x)^{n-1} \text{ for } x=1.$$

$$= n(1+1)^{n-1} = n \cdot 2^{n-1}.$$

$$(ii) \quad \frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \dots + \frac{n \cdot C_n}{C_{n-1}}$$

$$= \frac{n}{1} + 2 \cdot \frac{n(n-1)}{1 \cdot 2n} + \frac{3n(n-1)(n-2)2}{1 \cdot 2 \cdot 3n(n-1)} + \dots + \frac{n \times 1}{n}$$

$$= n + (n-1) + (n-2) + \dots + 1.$$

$$= \text{sum of the first } n \text{ natural numbers} = \frac{n(n+1)}{2}.$$

Ex. 2. Find the sum of the squares of the co-efficients in the expansion of $(1+x)^n$.

Here we have to find the value of

$$C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2;$$

$$\text{We have } (1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n. \quad (1)$$

Also by reversing the order

$$\begin{aligned} (1+x)^n &= C_nx^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \dots + C_0 \\ &= C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n \end{aligned} \quad (2)$$

because ${}^nC_r = {}^nC_{n-r}$

Multiplying (1) and (2) and equating the co-efficients of x^n on both sides we get

$$C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \text{co-efficient of } x^n \text{ in } (1+x)^n(1+x)^n$$

i.e., in $(1+x)^{2n}$.

$$\text{Hence the required sum} = {}^{2n}C_n = \frac{|2n|}{|n| |n|}.$$

(Ex. 1. Art 9.6)

Ex. 8. Find for what value of n the co-efficients of second, third and fourth terms of the expansion $(1+x)^n$ are in A. P.

Here the co-efficients of the second, third and fourth terms are ${}^nC_1$, ${}^nC_2$, ${}^nC_3$ i.e., n , $\frac{n(n-1)}{2}$, $\frac{n(n-1)(n-2)}{6}$.

They are in A. P. if $n(n-1) = \frac{1}{2}n(n-1)(n-2) + n$

$$\text{i.e., if } 6n - 6 = n^2 - 3n + 2 + 6 \quad [\because n \neq 0]$$

i.e., if $n^2 - 9n + 14 = 0$
 $n = 2$ or 7 .

But $n = 2$ means no 4th term. $\therefore n = 7$.

EXERCISE XL

1. In the expansion of $(a+x)^n$, what term from the beginning is :

- (i) the fifth term from the end ?
- (ii) the r th term from the end ?
- (iii) the $(r+1)$ th term from the end ?

2. If C_r denote the number of combinations of n things taken r at a time, show that

$$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots$$

Find the value of :

3. ${}^{14}C_1 + {}^{14}C_2 + {}^{14}C_3 + \dots + {}^{14}C_{14}$.

4. ${}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 + \dots + {}^{21}C_{21}$.

5. ${}^{18}C_1 + {}^{18}C_2 + {}^{18}C_3 + \dots + {}^{18}C_{17}$.

If $C_0, C_1, C_2, \dots, C_n$ be the co-efficients in the expansion of $(1+x)^n$, find the value of :

6. $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n$.

7. $C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + \dots + \frac{C_n}{n+1}$. (Q. 1947)

8. $C_0C_1 + C_1C_2 + C_2C_3 + \dots + C_{n-1}C_n$.

9. $C_0C_2 + C_1C_3 + C_2C_4 + \dots + C_{n-2}C_n$.

10. Find the r th term from the beginning and the r th term from the end of $(y+2x)^n$.

11. Find the $(n+1)$ th term from the end in the expansion of $\left\{x - \frac{1}{x}\right\}^{2n}$.

12. If p be the sum of the odd terms and q the sum of the even terms in the expansion of $(x+a)^n$ prove that

(i) $p^2 - q^2 = (x^2 - a^2)^n$.

(ii) $4pq = (x+a)^{2n} - (x-a)^{2n}$. (H.S.B ; P.U. ; D.U.)

13. Find the general term in the expansion of $(x+y)^n$ and the sum of the squares of the co-efficients in that of $(1+x)^n$, n being a positive integer.

*Greatest Co-efficients and Terms

*99. **Greatest Co-efficient.** Since the co-efficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$, it follows that we have to find the value of r for which ${}^nC_r$ is the greatest.

The various binomial co-efficients are

$${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \dots$$

or $1, n, n \cdot \frac{(n-1)}{2}, n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}, \dots$

The multiplying factors $n, \frac{n-1}{2}, \frac{n-2}{3}$ etc., which convert each co-efficient into the next, gradually decrease and the co-efficients themselves increase steadily only so long as the multiplying factors exceed 1, and begin to decrease steadily as soon as the multiplying factors become less than 1. But since ${}^nC_r = {}^nC_{n-r}$, i.e., the co-efficients equidistant from the beginning and end are equal, it follows that

(i) the greatest co-efficient is the middle one when n is even and is given by $r = \frac{1}{2}n$, and that

(ii) there are two equal greatest co-efficients when n is odd; these are two middle ones given by $r = \frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$.

Ex. 1. Find the greatest co efficient in the expansion of $(1+x)^{10}$.

Here $n=10$, therefore for the greatest co-efficient $r = \frac{1}{2}n = 5$, so that the greatest co-efficient is involved in the 6th term and is equal to ${}^{10}C_5 = 252$.

The same can be proved independently as follows :

Let the greatest co-efficient occur in T_{r+1} .

The co-efficient of T_{r+1} is ${}^{10}C_r$ and that of T_r is ${}^{10}C_{r-1}$

$$\therefore {}^{10}C_r \geq {}^{10}C_{r-1}.$$

$$\text{If } \frac{10 \cdot 9 \dots (10-r+2)(10-r+1)}{1 \cdot 2 \dots (r-1)r} \geq \frac{10 \cdot 9 \dots (10-r+2)}{1 \cdot 2 \dots (r-1)}$$

$$\text{i.e., if } \frac{10-r+1}{r} \geq 1$$

$$\text{i.e., if } 11 \geq 2r \text{ or } r \leq 5\frac{1}{2}.$$

Now as r cannot be equal to $5\frac{1}{2}$, therefore ${}^{10}C_r$ is never equal to ${}^{10}C_{r-1}$ and so long as $r < 5\frac{1}{2}$, ${}^{10}C_r > {}^{10}C_{r-1}$.

Hence up to the value $r=5$, the co-efficients continually increase, the greatest of them being given by $r=5$.

Ex. 2. Find the greatest co-efficient in the expansion of $(1+x)^{11}$.

Here $n=11$, therefore for the greatest co-efficient $r = \frac{1}{2}(11-1)$ or $\frac{1}{2}(11+1)$, i.e., 5 or 6; so that the greatest co-efficient occurs in the 6th or the 7th term and is equal to ${}^{11}C_5 = {}^{11}C_6 = 462$.

The same can be proved as follows :

The co-efficient of $T_{r+1} \geq$ co-efficient of T_r ,

$${}^{11}C_r \geq {}^{11}C_{r-1}$$

$$\text{i.e., if } \frac{11-r+1}{r} \geq 1$$

$$\text{i.e., if } 12 \geq 2r \text{ or } r \leq 6.$$

Therefore so long as $r < 6$, ${}^{11}C_r > {}^{11}C_{r-1}$ and when $r=6$, ${}^{11}C_{r-1} = {}^{11}C_r$ i.e., the co-efficients continually increase up to the value $r=6$ when they become equal. Therefore the greatest co-efficients occur in the 6th and 7th terms.

*9.91. **Greatest Term.** The method is the same as given in examples of the previous article.

Ex. Find the numerically greatest term in the expansion of $(2-3x)^7$ when $x=1$.

Here $T_{r+1} = {}^7C_r (2)^{7-r} (3x)^r$ (numerically)

and $T_r = {}^7C_{r-1} (2)^{7-r+1} (3x)^{r-1}$ (,)

Now $T_{r+1} \geq T_r$

$$\text{if } {}^7C_r (2)^{7-r} (3x)^r \geq {}^7C_{r-1} (2)^{7-r+1} (3x)^{r-1}$$

$$\text{i.e., if } \frac{7-r+1}{r} (3x) \geq 2$$

$$\text{i.e., if } 3(7-r+1) \geq 2r \text{ since } x=1$$

$$\text{i.e., if } 24 \geq 5r, \text{ or } r \leq 4\frac{4}{5}.$$

So that the terms continually increase up to $r=4$.

$\therefore$ the greatest term for $x=1$ is $T_5 = {}^7C_4 (2)^{7-4} (3x)^4 = 22680$.

EXERCISE XLI

1. Find the term or (terms) with the greatest co-efficients in the following :

$$(i) (1-3x)^6; (ii) (x-2y)^{12}; (iii) \left(x^2 + \frac{1}{x}\right)^{11}$$

2. Find the greatest term in the expansion of :

$$(i) (5x+4)^7 \text{ when } x=1, (ii) (1+x)^{10} \text{ when } x=\frac{2}{3}.$$

3. Find the numerically greatest term (or terms) in the expansion of

$$(i) (1-2x)^6 \text{ when } x=\frac{1}{4}; (ii) (2+3x)^7 \text{ when } x=\frac{1}{3};$$

$$(iii) \left(xy - \frac{1}{x}\right)^{11} \text{ when } x=1, y=2.$$

4. Find which is the greatest term in the binomial expansion of $(3+2x)^7$ when $x=\frac{1}{4}$.

CHAPTER X

BINOMIAL THEOREM

Any Index

10.1. We have already proved that when n is a positive integer, then

$$\begin{aligned}(1+x)^n &= {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n, \\ &= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots \\ &\quad + \frac{n(n-1)\dots(n-r+1)}{r}x^r + \dots + x^n.\end{aligned}$$

The process by which we established the above expansion clearly shows that it is true only when n is a positive integer. The case when n is not a positive integer or, in other words, when n is a negative integer or a positive or negative fraction has not been covered so far. We shall merely state what the theorem becomes in that case, the proof being beyond the scope of the present work.

10.2. *To state the Binomial Theorem when the index n is not a positive integer.*

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots \\ &\quad + \frac{n(n-1)\dots(n-r+1)}{r}x^r + \dots \text{ to infinity,}\end{aligned}$$

provided that the numerical value of x is less than unity.

Properties.

1. *It may be noted that this series extends to infinity. For in order that the series may stop at some stage $n-r+1$ must vanish, i.e., $r=n+1$, for some value of r .*

But r is always a positive integer. Hence $n+1$ or n must also be a positive integer if the series is to stop. We concluded therefore, that when n is not a positive integer, the above expansion extends to infinity.

2. The general term in the expansion of $(1+x)^n$ is $\frac{n(n-1)\dots(n-r+1)}{r!} x^r$ and not ${}^nC_r x^r$, because when n is not a positive integer, ${}^nC_r$ has no significance.

3. The condition that the numerical value of x must be less than one is absolutely essential. For, if it is not so, the series is not convergent and absurd results follow. For example, expanding $(1-x)^{-2}$ by Binomial Theorem, we get

$$\begin{aligned}(1-x)^{-2} &= 1 + (-2)(-x) + \frac{(-2)(-2-1)}{1 \cdot 2}(-x)^2 + \dots \\ &= 1 + 2x + 3x^2 + \dots\end{aligned}$$

Putting x equal to 2, we have

$$(-1)^{-2} = 1 = 1 + 2 \cdot 2 + 3 \cdot 2^2 + \dots$$

which is evidently absurd.

Similarly whenever x is numerically equal to or greater than 1, the expansion may be invalid.

10.3. Expansion of $(1+x)^{-n}$. To expand $(1+x)^{-n}$ and to find the general term in the expansion $(-1 < x < 1)$.

By the above formula

$$\begin{aligned}(1+x)^{-n} &= 1 + (-n)x + \frac{(-n)(-n-1)}{2}x^2 + \dots \\ &= 1 - nx + \frac{n(n+1)}{2}x^2 - \dots\end{aligned}$$

The general term T_{r+1}

$$\begin{aligned}&= \frac{(-n)(-n-1)\dots(-n-r+1)}{r!} x^r \\ &= \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} (-1)^r x^r.\end{aligned}$$

Cor. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2}x^2 + \dots$ the general

term being $\frac{n(n+1)\dots(n+r-1)}{r!} (-1)^r (-x)^r$

i.e., $\frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r$. [$\because (-1)^{2r} = 1$].

Note.—It may be observed that when x is positive and n is negative, the terms are alternately positive and negative but when x and n are both negative, then the terms are all positive.

The following particular cases of the above expansions which can be easily deduced should be remembered :

$$(1-x)^{-1} = 1 + x + x^2 + \dots + x^r + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots + (r+1)x^r + \dots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + \dots + \frac{(r+1)(r+2)}{2}x^r + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - \dots + (-1)^r x^r + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - \dots + (-1)^r (r+1)x^r + \dots$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - \dots + (-1)^r \frac{(r+1)(r+2)}{2}x^r + \dots$$

10.4 Expansion of $(x+a)^n$. When the index n is not a positive integer, it is desirable to reduce every binomial to a form in which the first term is unity and the second term numerically less than unity. Thus $(a+x)^n$ should be put in the form of $a^n \left(1 + \frac{x}{a}\right)^n$ or $x^n \left(1 + \frac{a}{x}\right)^n$ according as x is numerically less than or greater than a , i.e., according as x/a is numerically less than or greater than 1.

EXAMPLES

Ex. 1. Expand to four terms the expression $(3-x)^{-2}$ when x is positive and less than 3.

$$(3-x)^{-2} = 3^{-2} \left(1 - \frac{x}{3}\right)^{-2} = \frac{1}{9} \left(1 - \frac{x}{3}\right)^{-2}$$

Here x being less than 3, $\frac{x}{3}$ is less than 1, and therefore expanding it by the Binomial Theorem, we get,

$$(3-x)^{-2} = \frac{1}{9} \left[1 + (-2) \left(-\frac{x}{3}\right) + \frac{(-2)(-2-1)}{1.2} \left(-\frac{x}{3}\right)^2 + \frac{(-2)(-2-1)(-2-2)}{1.2.3} \left(-\frac{x}{3}\right)^3 + \dots \right]$$

$$= \frac{1}{9} \left[1 + \frac{2}{3}x + \frac{1}{3}x^2 + \frac{4}{27}x^3 + \dots \right]$$

$$= \frac{1}{9} + \frac{2}{27}x + \frac{1}{27}x^2 + \frac{4}{243}x^3 + \dots$$

Ex. 2. Use the Binomial Theorem to expand $(1-x+x^2)^4$ in ascending powers of x . (P. U. 1944)

$$(1-x+x^2)^4 = \left(\frac{1+x^3}{1+x} \right)^4 = (1+x^3)^4 (1+x)^{-4}$$

$$= (1+4x^3+6x^6+4x^9+x^{12}) \times$$

$$\left[1 - 4x + \frac{-4(-5)}{2}x^2 + \frac{(-4)(-5)(-6)}{3}x^3 + \dots \right]$$

$$= (1+4x^3+6x^6+4x^9+x^{12}) [1-4x+10x^2-20x^3+35x^4+\dots]$$

$$= (1-4x+10x^2-20x^3+35x^4+\dots) + (4x^3-16x^4+\dots)$$

$$= 1-4x+10x^2-16x^3+19x^4+\dots$$

Ex. 3. Find the first negative term in the expansion of $(1+3x)^{\frac{10}{3}}$, where x is positive and less than $\frac{1}{3}$.

Let T_{r+1} be the first negative term, so that T_r is positive.

In the expansion of $(1+x)^n$, we have

$$T_{r+1} = \frac{n(n-1)\dots(n-r+1)}{r!} x^r$$

$$= \frac{n(n-1)\dots(n-r+2)}{(r-1)!} x^{r-1} \cdot \frac{n-r+1}{r} x$$

$$= T_r \cdot \frac{n-r+1}{r} x$$

$\therefore T_{r+1}$ is the first negative term when $n-r+1$ is first negative for the smallest value of r .

$\therefore$ In this case $\frac{10}{3} - r + 1 = 4\frac{1}{3} - r$ is first negative for the smallest value of r , when r is an integer just $> 4\frac{1}{3}$.

Hence when $r=5$, we get the first negative term i.e., the sixth term is the first negative term which is :

$$\begin{aligned} &= \frac{\frac{10}{3} \left(\frac{10}{3} - 1 \right) \left(\frac{10}{3} - 2 \right) \left(\frac{10}{3} - 3 \right) \left(\frac{10}{3} - 4 \right) (3x)^5}{5!} \\ &= -\frac{10 \times 7 \times 4 \times 1 \times 2}{5 \times 4 \times 3 \times 2} x^5 = -\frac{14}{3} x^5. \end{aligned}$$

EXERCISE XLII

1. Expand to 4 terms :

$$(i) (x-a)^{\frac{2}{3}} (x>a). \quad (ii) (a-x^4)^{\frac{3}{4}} (x^4<a).$$

$$(iii) \left(x - \frac{1}{x}\right)^{-1} (x>1). \quad (iv) (1-2x)^{-\frac{1}{2}} (x<\frac{1}{2}).$$

2. Find the cube root of $(8+x)^4$ to four terms when
(i) $x<8$. (ii) $x>8$.

3. Find the co-efficient of x^{12} in the expansion of

$$\frac{a^3}{(a^3-x^3)^{\frac{2}{3}}}.$$

4. Find the general term of $(1+2x)^{-\frac{1}{2}}$. When is the expansion valid ?

5. Find the first negative term in the expansion of

$$(i) (1+3x)^{\frac{16}{3}}.$$

$$(ii) (1+5x^2)^{\frac{19}{5}}.$$

$$(iii) (1+x)^{\frac{10}{3}}.$$

$$(v) (1+\frac{2}{3}x)^{\frac{13}{4}}.$$

6. Expand $(1-x)^{-1}$ into a series by the Binomial Theorem. When is the expansion valid ? Check the validity of the expansion when $x=\frac{1}{2}$, 2. (K. U. 1951)

Applications

10.5. To apply the Binomial Theorem in finding approximate values of certain expressions.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{6} x^3 + \dots$$

Now if x is so small that its second and higher powers can be neglected, then obviously from above $(1+x)^n = 1 + nx$, approximately

But if x^2 is not negligible and the other higher powers of x are negligible, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2, \text{ nearly ; and so on.}$$

It is clear that the larger the number of terms taken the better the approximation.

Ex. 1. When x is small show that

$$\frac{(1+x)^{\frac{1}{2}} + (1-2x)^{\frac{1}{4}}}{(1+3x)^{\frac{1}{6}} + (1+5x)^{\frac{1}{10}}} = 1 - \frac{x}{2} \text{ nearly.}$$

Neglecting square and higher powers of x , we have

$$\begin{aligned} \frac{(1+x)^{\frac{1}{2}} + (1+2x)^{\frac{1}{2}}}{(1+3x)^{\frac{1}{2}} + (1+5x)^{\frac{1}{2}}} &= \frac{(1+\frac{1}{2}x) + (1-\frac{1}{4}x)}{(1+\frac{3}{8}x) + (1+\frac{5}{16}x)} = \frac{2+\frac{1}{2}x-\frac{1}{4}x}{2+\frac{3}{8}x+\frac{5}{16}x} \\ &= \frac{2}{2+x} = \frac{1}{\left(1+\frac{x}{2}\right)} = \left(1+\frac{x}{2}\right)^{-1} \\ &= 1 - \frac{x}{2} \text{ nearly.} \end{aligned}$$

Ex. 2. If x be so small that its squares and higher powers are neglected, show that

$$\frac{(9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{4}}}{4+5x} = \frac{1}{4} - \frac{17x}{384}$$

$$\text{L. H. S.} = \frac{3\left(1+\frac{7x}{9}\right)^{\frac{1}{2}} - 2\left(1+\frac{3}{16}x\right)^{\frac{1}{4}}}{4\left(1+\frac{5x}{4}\right)}$$

$$= \frac{1}{4} \left[3 \left(1+\frac{7x}{18}\right) - 2 \left(1+\frac{3}{64}x\right) \right] \left(1+\frac{5}{4}x\right)^{-1}$$

$$= \frac{1}{4} \left(3 + \frac{7x}{6} - 2 - \frac{3}{32}x \right) \left(1 - \frac{5x}{4} \right)$$

neglecting x^2 and higher powers,

$$= \frac{1}{4} \left(1 + \frac{103}{96}x \right) \left(1 - \frac{5x}{4} \right)$$

$$= \frac{1}{4} \left(1 + \frac{103}{96}x - \frac{5}{4}x \right) = \frac{1}{4} - \frac{17x}{384}.$$

$$\text{Ex. 3. Show that } \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} = \left[\frac{p}{q} \right]^{\frac{1}{n}}.$$

approximately when p is nearly equal to q .

(P. U. 1950)

Let $p = q + h$, so that h is small.

$$\begin{aligned}
\frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} &= \frac{(n+1)(q+h) + (n-1)q}{(n-1)(q+h) + (n+1)q} = \frac{2nq + (n+1)h}{2nq + (n-1)h} \\
&= \left(1 + \frac{n+1}{2n} \cdot \frac{h}{q} \right) \left(1 + \frac{n-1}{2n} \cdot \frac{h}{q} \right)^{-1} \\
&= \left(1 + \frac{n+1}{2n} \cdot \frac{h}{q} \right) \left(1 - \frac{n-1}{2n} \cdot \frac{h}{q} + \dots \right) \\
&= 1 + \frac{n+1}{2n} \cdot \frac{h}{q} - \frac{n-1}{2n} \cdot \frac{h}{q} + \dots \\
&= 1 + \frac{1}{n} \cdot \frac{h}{q} + \dots = 1 + \frac{1}{n} \cdot \frac{h}{q} \text{ nearly.}
\end{aligned}$$

Also $\left(\frac{p}{q} \right)^{\frac{1}{n}} = \left[1 + \frac{h}{q} \right]^{\frac{1}{n}} = 1 + \frac{1}{n} \cdot \frac{h}{q} \text{ nearly.}$

Hence neglecting the square and the higher powers of $(p-q)$, we have

$$\frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} = \left[\frac{p}{q} \right]^{\frac{1}{n}} \text{ approximately.}$$

Ex. 4. Find the cube root of 999 to 4 places of decimals.

$$\begin{aligned}
(999)^{\frac{1}{3}} &= (1000-1)^{\frac{1}{3}} = \left[1000 \left[1 - \frac{1}{1000} \right] \right]^{\frac{1}{3}} \\
&= (1000)^{\frac{1}{3}} (1 - .001)^{\frac{1}{3}} \\
&= 10 \left[1 - \frac{1}{3}(.001) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2}(-.001)^2 \right. \\
&\quad \left. + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3}(-.001)^3 + \dots \right] \\
&= 10 \left[1 - \frac{.001}{3} - \frac{1}{6}(.000001) - \frac{1}{54}(.00000001) + \dots \right] \\
&= 10 \{ 1 - .00033 - .0000001 - \dots \} = 10[1 - .00033] \\
&= 9.9967.
\end{aligned}$$

Note — We neglect the third term and onwards because they are too small to effect the result up to 4 places of decimals.

Ex. 5. Find the cube root of 126 to 5 places of decimals.

$$(126)^{\frac{1}{3}} = (125+1)^{\frac{1}{3}} + [125(1+\frac{1}{125})]^{\frac{1}{3}} = (1+0.008)^{\frac{1}{3}}$$

$$= 5 \left[1 + \frac{1}{3} \times 0.008 + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2} (0.008)^2 + \dots \right]$$

$$= 5[1 + 0.0026666 - \frac{1}{6} \times 0.000064 + \dots]$$

$$= 5[1.0026666 - 0.000007 + \dots]$$

$$= 5.013333 - 0.000035 = 5.013298$$

$$= 5.01330, \text{ correct to five decimal places.}$$

Ex. 6. *Approximate division.* — Divide 4831 by 996.

$$4831 \div 996 = 4831 \div (1000 - 4) = 4831 \div 1000 \left(1 - \frac{4}{1000} \right)$$

$$= 4.831(1 - 0.004)^{-1}$$

$$= 4.831[1 + 0.004 + 0.000016 + \dots]$$

$$= 4.831 + 0.019324 + \dots$$

$$= 4.850$$

a result which is correct to 3 decimal places.

EXERCISE XLIII

When x is so small that its square and higher powers may be neglected find the approximate value of

1. $(8+4x)^{\frac{1}{3}} (16-x)^{\frac{1}{4}}$

2. $\frac{\left(1 - \frac{2}{3}x\right)^{-3} + (1+5x)^{-\frac{1}{2}}}{(1-x)^{\frac{3}{2}}}$

3. $\frac{(1+2x)^{\frac{1}{3}} + (1-3x)^{\frac{1}{2}}}{(1+5x)^{\frac{1}{5}} + (1+3x)^{\frac{1}{3}}}$

4. If x be so small that its square and higher powers can be neglected find the value of :

(i) $\frac{(1-3x)^{\frac{1}{2}} + (1-x)^{\frac{5}{8}}}{(4-x)^{\frac{1}{2}}}$

(Q. 1944)

(ii) $\frac{(1+x)^{\frac{1}{2}} + (1-x)^{\frac{2}{3}}}{(1+x)^{\frac{2}{3}} + (1-x)^{\frac{1}{2}}}$

(Q. 1946)

5. $\frac{\sqrt{1-3x} + 3\sqrt{(1-x)^5}}{\sqrt{4-x}}$

6. Use the Binomial Theorem to find the value correct to 3 decimal places of :

- (i) $5758 \div 997$. (ii) $7.821 \div 1.002$.
 (iii) $5.231 \div (0.996)^2$.

7. If the square and higher powers of the difference between x and unity can be neglected, show that

$$mx^m - nx^n = (m-n)x^{m+n} \text{ nearly} \quad (\text{P. U. 1925})$$

8. If the square and higher powers of the difference between m and M can be neglected, show that

$$\sqrt{\frac{M}{m}} = \frac{M}{m+M} + \frac{m+M}{4m} \text{ nearly.}$$

[Hint. Put $M = m + h$, so that squares and higher powers of h can be neglected. The left-hand side is then

$$\left(1 + \frac{h}{m}\right)^{\frac{1}{2}} = 1 + \frac{h}{2m}. \quad \text{Similarly simplify the right-hand side.}$$

9. The side of a cubical block is one ft. at 40°C . When the block is heated to 41°C , the side becomes 1.01 ft. Find approximately the cubical expansion.

10. A cubical block is immersed in water and it is observed that the water displaced is 999 c.c. If the actual volume of the cube is 1000 c.c. find the experimental error in calculating the side of the cube correct to three decimal places.

11. If x be so small that its square and higher powers are neglected, then

$$\frac{(9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{2}}}{4+5x} = \frac{1}{4} - \frac{17x}{384}. \quad (\text{P. U.})$$

12. Evaluate $\left[1 - \frac{3}{10}\right]^{-\frac{1}{2}}$ correct to 3 places of decimals.

13. Find the value of

$$(i) \sqrt[5]{3128}. \quad (ii) \left[1 + \frac{3}{10}\right]^{\frac{1}{5}}. \quad (iii) (.998)^{-\frac{1}{5}}.$$

correct to 5 places of decimals.

14. By the use of a binomial expansion evaluate $(1.006)^{12}$ correct to 5 places of decimals.

15. Find the value of $(1.998)^7$ correct to 4 decimal places.

16. Find the cube roots of 1002 and 126 correct to 4 places of decimals.

17. By using the expression $\frac{7}{4} \left(1 - \frac{1}{49} \right)^{\frac{1}{2}}$; show that the value of $\sqrt{3}$ is approximately $\frac{19011}{10976}$.

[Hint. Take first three terms of the expansion.]

18. Obtain the value of the cube root of 2, correct to five places of decimals from $\frac{5}{4} \left(1 + \frac{3}{125} \right)^{\frac{1}{3}}$. (P. U. 1928)

19. Apply result of Ex. 3 page 173 to get approximately the seventh root of $\frac{131}{133}$. (P. U. 1950)

20. Find to three places of decimals the cube root of 998. (H. S. B. 1951)

*Further Examples on Binomial Theorem

Ex. 1. From what term will the terms of $(1-x)^{\frac{9}{5}}$ have the same sign, x being positive and < 1 ?

In the expansion of $(1-x)^n$, we have

$$T_{r+1} = \frac{n-r+1}{r} (-x) T_r.$$

The two successive terms T_r , T_{r+1} have the same sign when $\frac{n-r+1}{r} (-x)$ is positive. Since x and r are positive it requires that $n-r+1$ should be negative, so that

$$\frac{9}{5} - r + 1 < 0 \text{ or } r > \frac{14}{5}.$$

i.e. r is an integer just greater than $2\frac{4}{5}$.

Therefore the third and all the subsequent terms shall have the same sign.

Also as $T_3 = \frac{\frac{9}{5}(\frac{9}{5}-1)}{1 \cdot 2} (-x)^2$, a positive quantity; therefore the sign of the third and subsequent terms shall be positive.

Ex. 2. Find the co-efficient of x^n in the expansion of $\frac{(1+x)^2}{(1-x)^2}$, (x is +ve and < 1 .),

$$(1+x)^2 = 1 + 2x + x^2$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots + (r+1)x^r + \dots$$

$$= 1 + p_1x + p_2x^2 + \dots + p_rx^r + \dots \text{ say where } p_r = r+1.$$

$$\therefore (1+x)^2(1-x)^{-2} = (1+2x+x^2)(1+p_1x+p_2x^2+\dots)$$

Co-efficient of x^n

$$= p_n + 2p_{n-1} + p_{n-2}, \quad p_n \text{ being the co-efficient of } x^n \text{ in } (1-x)^{-2}$$

$$= (n+1) + 2n + n - 1 = 4n.$$

Ex. 3. Identify the series $1 + \frac{15}{8} + \frac{15}{8} \cdot \frac{21}{16} + \frac{15}{8} \cdot \frac{21}{16} \cdot \frac{27}{24}$

+ ... as a binomial expansion and hence find its sum.

Let the given series be identical with $(1+x)^n$ i.e.,

$$1 + nx + \frac{n(n-1)x^2}{2} + \dots$$

Comparing this with the given series, we get $nx = \frac{15}{8}$ (i)

$$\text{and } \frac{n(n-1)}{2} x^2 = \frac{15}{8} \cdot \frac{21}{16} \quad \dots\dots(ii)$$

Dividing (ii) by the square of (i) we have

$$\frac{n(n-1)x^2}{2n^2x^2} = \frac{15}{8} \cdot \frac{21}{16} \left(\frac{8}{15}\right)^2$$

$$\text{or } \frac{n-1}{2n} = \frac{21}{16} \cdot \frac{8}{15} = \frac{7}{2 \times 5} \text{ or } n-1 = \frac{7}{5}n \therefore n = -\frac{5}{2}.$$

$$\text{From (i) } -\frac{5}{2}x = \frac{15}{8} \therefore x = -\frac{3}{4}.$$

Hence the given series is the expansion of $\left(1 - \frac{3}{4}\right)^{-\frac{5}{2}}$

and can be written as

$$1 + \left(-\frac{5}{2}\right)\left(-\frac{3}{4}\right) + \frac{-\frac{5}{2}\left(-\frac{5}{2}-1\right)}{2}\left(-\frac{3}{4}\right)^2 + \dots$$

$$\text{and its sum is } \left(1 - \frac{3}{4}\right)^{-\frac{5}{2}} = (4)^{\frac{5}{2}} = 32.$$

Ex. 4. Show that

$$1 + \frac{1}{3} + \frac{1.3}{2}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3}\left(\frac{1}{3}\right)^3 + \dots \text{ to infinity}$$

is equal to $\sqrt{3}$.

(P. U.)

Let the given series be the expansion of $(1+x)^n$, where n is not a positive integer.

We have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{3} x^3 + \dots$$

Comparing this with the given series, we get

$$nx = \frac{1}{3} \quad (1)$$

$$\text{and } n(n-1)x^2 = 1.3 \left(\frac{1}{3}\right)^2 = \frac{1}{3} \quad (2)$$

Squaring (1) and dividing the result by (2), we have

$$\frac{n}{n-1} = \frac{1}{3}, \text{ i.e., } 2n = -1 \text{ or } n = -\frac{1}{2}.$$

$$\therefore \text{ from (1) } x = -\frac{2}{3}.$$

Hence the given series is the expansion of $\left(1 - \frac{2}{3}\right)^{-\frac{1}{2}}$

$$\text{which becomes } = \left(\frac{1}{3}\right)^{-\frac{1}{2}} = (3)^{\frac{1}{2}} = \sqrt{3}.$$

Ex. 5. If $y = \frac{1}{3} + \frac{1.3}{1.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \dots \infty$,
prove that $y^2 + 2y - 2 = 0$. (P.U.)

Adding 1, we find that $y+1$ is equal to the series given in Ex. 4 which has been proved equal to $\sqrt{3}$.

$$\therefore y+1 = \sqrt{3} \text{ or } (y+1)^2 = 3 \therefore y^2 + 2y - 2 = 0.$$

Note. It may be remarked that all series cannot be identified as binomial expansions.

EXERCISE XLIV

1. From which term will the terms of the following expansions have the same sign? Give the sign in each case.

$$(i) (1-x)^{\frac{7}{5}}, \quad (ii) (1-x)^{-5}, \quad (iii) (1-x)^{\frac{35}{9}}.$$

2. Find the co-efficient of x^n in the expansion of

$$(i) \frac{(1+x)^3}{(1-x)^2}, \quad (ii) \frac{(1+x)^2}{(1-x)^3}. \quad (\text{P.U.})$$

2. Write down the expansion of $(1+x)^{2n}$ where n is a positive integer and show that the sum of the co-efficients of the odd terms is 2^{2n-1} .

3. In the expansion of $(1+x)^{m+n}$, where m and n are positive integers, prove that the co-efficients of x^m and x^n are equal.

Obtain the first four terms in the expansion of $(1-4x)^{\frac{1}{2}}$.

4. Find the term independent of x in the expansion of $\left(x + \frac{1}{x}\right)^{10}$.

5. Write down the first four terms in the expansion of $(1+x)^{\frac{1}{2}}$ in ascending powers of x .

6. Find the co-efficient of x^7 in $(1-5x)^{-\frac{3}{4}}$.

7. Find in its simplest form the general term of $(1-x)^{-4}$.
Find the co-efficient of x^4 in the expansion of $(1-x+x^2-x^3)^{-1}$.

8. If in the expansion of $(1+x)^{2n+1}$ the co-efficients of x^r and x^{n+1} are equal, find r .

9. Show that

$$(1+x)^{-n} = \frac{1}{2^n} \left\{ 1 + n \left[\frac{1-x}{1+x} \right] + \frac{n(n-1)}{2} \left[\frac{1-x}{1+x} \right]^2 + \dots \right\}.$$

10. Find the co-efficient of x^5 in the expansion of $\frac{1+x}{1-x}$.

11. Prove that the co-efficient of x^n in the expansion of $\left(\frac{1+x}{1-x}\right)^2$ is $4n$. ($x < 1$).

Point out an exceptional case, if any.

12. Find the term independent of x in $\left(3x^2 - \frac{2}{x^3}\right)^{20}$.

Find the co-efficient of x^n in the expansion of

$$\frac{3x^2-2}{x+x^2}.$$

13. State the Binomial Theorem for a negative or fractional index.

11. Prove that :

$$(i) \quad \frac{7}{5} \left(1 + \frac{1}{10^2} + \frac{1.3}{1.2} \cdot \frac{1}{10^4} + \frac{1.3.5}{1.2.3} \cdot \frac{1}{10^6} + \dots \right) = \sqrt{2}.$$

$$(ii) \quad 1 + \frac{1}{4} + \frac{1.4}{2} \left(\frac{1}{4} \right)^2 + \frac{1.4.7}{3} \left(\frac{1}{4} \right)^3 + \dots = (4)^{\frac{1}{3}}.$$

$$12. \quad \text{Show that } \sqrt[3]{3} = 1 + \frac{2}{3} - \frac{4}{9} + \frac{40}{81} - \frac{160}{243} + \dots$$

13. If x be so small that its cube and higher powers may be neglected, prove that

$$(i) \quad \frac{(1-x)^{-\frac{5}{2}} + (16+8x)^{\frac{1}{2}}}{(1+x)^{-\frac{1}{2}} + (2+x)^2} = 1 + \frac{23}{40}x^2. \quad (\text{P. U.})$$

$$(ii) \quad \frac{3 \left(x + \frac{4}{9} \right)^{\frac{1}{2}} \left(1 - \frac{3}{4}x^2 \right)^{\frac{1}{3}}}{2 \left(1 + \frac{9}{16}x \right)^2} = 1 - \frac{307}{256}x^2.$$

14. If x be nearly equal to 1, show that

$$\frac{ax^b - bx^a}{x^b - x^a} = \frac{1}{1-x} \text{ nearly. } (\text{Q. 1951})$$

REVISION QUESTIONS IV

1. (a) Expand $(1-x)^n$ in powers of x when n is a negative integer. Can we put $n=-1$, $x=2$ on both sides of the identity?

(b) Find the co-efficient of x^6 in the expansion of $(1+x+x^2)^{-3}$.

(a) Sol. We have

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1 \cdot 2}x^2 - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \dots$$

Here $n=-1$ is possible but x cannot be 2. The reason for the latter case is that the series on the right hand side for $x=2$ is not convergent. When $x=2$ and $n=-1$ every term on the right-hand will be seen to be positive whereas left-hand side $= (1-2)^{-1} = -1$. The equality therefore will become absurd.

$$(b) \quad [\text{Hint : } (1+x+x^2)^{-3} = \left(\frac{1-x}{1-x^3} \right)^3 = (1-x)^3 (1-x^3)^{-3} \\ = (1-3x+3x^2-x^3)(1-x^3)^{-3}$$

the co-efficient of x^6 can be seen now to be 3.

CHAPTER XI

PARTIAL FRACTIONS

11.1. The student is already familiar with the process of simplifying a group of fractions connected by the signs of addition and subtraction to one single fraction. We shall now deal with the converse process of decomposing a given fraction into a group of simpler fractions each having for its denominator one of the factors of the denominator of the original fraction. These simpler fractions are called Partial Fractions. For example

$$\frac{2x-3}{(x-1)(x-2)} = \frac{1}{x-1} + \frac{1}{x-2};$$

$\frac{1}{x-1}$, $\frac{1}{x-2}$ are partial fractions of $\frac{2x-3}{(x-1)(x-2)}$.

11.2. For the sake of brevity and convenience we shall introduce the following notation:

We shall denote any one given algebraical expression of x by $f(x)$, and its value for $x=a$ by $f(a)$. For example; if x^2-2x+5 is a given algebraical expression, we can denote it by $f(x)$ and say that $f(x)=x^2-2x+5$; the value of the expression for $x=3$ shall be denoted by $f(3)$, i.e., $f(3)=3^2-2 \times 3+5=8$. Similarly if there be two different algebraical expressions, we can denote one of them by $f(x)$ and the other by $F(x)$. Thus $f(x)$ and $F(x)$ will be two different algebraical expressions. Similarly suitable notations may be used whenever we have to deal with more than two expressions.

Ex. 1. Show that if $f(x) = \frac{x^2-2x+3}{x+2}$,

then $f(3) = \frac{6}{5}$ and $f(4) = \frac{11}{6}$.

Ex. 2. If $f(x) = 3x^3+1$ and $F(x) = \frac{2x^3+3}{x+1}$,

then $f(1) = 4$, and $F(1) = \frac{5}{2}$.

Ex. 3. If $f(x) = x^2+x+1$, show that $f(1+y) = 3+3y+y^2$.

Ex. 4. If $f(y) = 1+y^2$, show that $f(x-1) = 2-2x+x^2$.

11.3. *To find out the partial fractions when the denominator consists of only linear non-repeated factors.*

Whenever a given fraction is to be broken up into partial fractions, the first essential step is to see whether the degree of the numerator is lower than that of the denominator. In case the degree of the numerator is not less than that of the denominator, the numerator must be divided out by the denominator till the remainder is of lower degree than the denominator. Thus the given fraction reduces to a simple quotient and a fraction whose numerator is of lower degree than the denominator. The quotient is left as it is and the remaining fraction is broken up into partial fractions.

Ex. 1. Resolve $\frac{x^2+2x+4}{(x-1)(x-2)}$ into partial fractions.

Here the numerator is not of lower degree than the denominator. Therefore dividing out, we have

$$\frac{x^2+2x+4}{(x-1)(x-2)} = \frac{x^2+2x+4}{x^2-3x+2} = 1 + \frac{5x+2}{(x-1)(x-2)}.$$

Now we shall break $\frac{5x+2}{(x-1)(x-2)}$ into partial fractions.

$$\text{Let } \frac{5x+2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}.$$

$$\therefore 5x+2 = A(x-2) + B(x-1).$$

In this identity putting $x=2$, we get $B=12$.

Also putting $x=1$, we get $A=-7$.

$$\therefore \frac{x^2+2x+4}{(x-1)(x-2)} = 1 - \frac{7}{x-1} + \frac{12}{x-2}.$$

Note. It may be observed that if we suppress the factor $x-2$ from the fraction and then put $x=2$ we get 12 which is the value of B . Similarly we get the value of A .

Ex. 2. Resolve $\frac{x-2}{6x^2-7x+2}$ into partial fractions.

We have

$$\frac{x-2}{6x^2-7x+2} = \frac{x-2}{(2x-1)(3x-2)}$$

$$\text{Let } \frac{x-2}{(2x-1)(3x-2)} = \frac{A}{2x-1} + \frac{B}{3x-2} = \frac{A(3x-2) + B(2x-1)}{(2x-1)(3x-2)}.$$

$$\text{Then } x-2 = A(3x-2) + B(2x-1)$$

Putting $x = \frac{1}{2}$, we have

$$\frac{1}{2} - 2 = A\left(\frac{3}{2} - 2\right) = -\frac{1}{2}A. \quad \therefore A = 3$$

Also putting $x = \frac{2}{3}$ we have

$$\frac{2}{3} - 2 = B\left(\frac{4}{3} - 1\right) = \frac{1}{3}B, \quad \therefore B = -4.$$

$$\therefore \frac{x-2}{6x^2-7x+2} = \frac{x-2}{(2x-1)(3x-2)} = \frac{3}{2x-1} - \frac{4}{3x-2}.$$

Ex. 3. Resolve into partial fractions $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$.

Here the degree of the numerator is the same as that of the denominator and therefore we have

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{A}{x-4} + \frac{B}{x-5} + \frac{C}{x-6}.$$

$$\therefore (x-1)(x-2)(x-3) = (x-4)(x-5)(x-6) + A(x-5)(x-6) + B(x-4)(x-6) + C(x-4)(x-5).$$

Putting $x=4$, we get

$$3 \times 2 \times 1 = A(-1)(-2) \quad \text{i.e. } A = 3$$

Putting $x=5$, we have

$$4 \times 3 \times 2 = B.1(-1) \quad \therefore B = -24$$

Again putting $x=6$, we have

$$5 \times 4 \times 3 = C \times 2 \times 1 \quad \therefore C = 30$$

$$\text{Hence the given fraction} = 1 + \frac{3}{x-4} - \frac{24}{x-5} + \frac{30}{x-6}.$$

***11.4. General Case.** To find out the partial fraction corresponding to a linear non-repeated factor in the denominator.

Let the given fraction be

$$\frac{f(x)}{(x-a)F(x)},$$

where $F(x)$ does not contain $(x-a)$ as a factor and consequently $F(x)$ does not vanish for $x=a$ and when the degree of the numerator is lower than that of the denominator.

$$\text{Now let } \frac{f(x)}{(x-a)F(x)} = \frac{A}{x-a} + \frac{R(x)}{F(x)}.$$

A being independent of x .

$$\text{Hence } \frac{f(x)}{(x-a)F(x)} = \frac{AF(x) + (x-a)R(x)}{(x-a)F(x)}$$

$$\text{or } f(x) = AF(x) + (x-a)R(x).$$

This is an identity and therefore true for all values of the variable x ; putting $x=a$, we get

$$f(a) = AF(a) \text{ i.e., } A = \frac{f(a)}{F(a)}.$$

$\therefore$ the partial fraction corresponding to $(x-a)$ is $\frac{A}{(x-a)}$,

$$\text{where } A = \frac{f(a)}{F(a)}.$$

Hence the rule to find A is :

Suppress the factor $(x-a)$ from the fraction and put $x=a$ in every portion of the fraction.

Note.—The partial fraction corresponding to the non-repeated linear factor $(x-a)$ in the denominator is $\frac{A}{(x-a)}$, where A is independent of x .

$$\begin{aligned} \text{Ex. 1. } & \frac{x}{(x+1)(x-1)(x+2)} \\ &= \frac{-1}{(x-1)(-2) \cdot 1} + \frac{1}{2 \cdot (x-1) \cdot 3} + \frac{-2}{(-1)(-3)(x+2)} \\ &= \frac{1}{2(x+1)} + \frac{1}{6(x-1)} - \frac{2}{3(x+2)}. \end{aligned}$$

$$\text{Ex. 2. Put into partial fractions } \frac{x^2}{(x-1)(x-2)}.$$

Here the numerator is not of lower degree than the denominator. Therefore dividing out, we have

$$\begin{aligned} \frac{x^2}{(x-1)(x-2)} &= 1 + \frac{3x-2}{x^2-3x+2} = 1 + \frac{3x-2}{(x-1)(x-2)} \\ &= 1 + \frac{3-2}{(x-1)(-1)} + \frac{6-2}{1 \cdot (x-2)} = 1 - \frac{1}{x-1} + \frac{4}{x-2}. \end{aligned}$$

Ex. 3. Put into partial fractions

$$\frac{x^2}{(x^2+1)(x^2+4)}.$$

Here since all powers of x are even, let us put $x^2=z$, so that the given expression becomes

$$\frac{z}{(z+1)(z+4)}.$$

We, now proceed to put this into partial fractions.

Let
$$\frac{z}{(z+1)(z+4)} = \frac{A}{z+1} + \frac{B}{z+4} = \frac{A(z+4) + B(z+1)}{(z+1)(z+4)}$$

$$\therefore z = A(z+1) + B(z+1).$$

In this identity comparing like powers of z on both the sides we get

$$\begin{aligned} 1 &= A + B, & 0 &= 4A + B \\ \therefore 1 &= -3A \text{ or } A = -\frac{1}{3} \text{ and } B = -4A = \frac{4}{3}, \end{aligned}$$

so that
$$\frac{z}{(z+1)(z+4)} = -\frac{1}{3} \frac{1}{z+1} + \frac{4}{3} \frac{1}{z+4}.$$

Hence
$$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3} \frac{1}{x^2+1} + \frac{4}{3} \frac{1}{x^2+4} = \frac{1}{3} \left(\frac{4}{x^2+4} - \frac{1}{x^2+1} \right).$$

EXERCISE XLV

Decompose into partial fractions :

1. $\frac{x+1}{(2-x)(x-3)}$ (H. S. 1951).

2. $\frac{7x+5}{3x^2+4x+1}$.

8. $\frac{x^3+3x-3}{(x-1)(x+2)}$.

4. $\frac{3-4x-5x^2-x^3}{(x+3)(x+2)}$.

5. $\frac{1+x+2x^3}{x(x+1)(2x+1)}$.

6. $\frac{3x^2-5x+1}{x^2-2x-3}$.

7. $\frac{2x^2+1}{2x^2+3x+1}$.

8. $\frac{2x^3+7x^2-2x-2}{2x^2+x-6}$.

9. $\frac{x-2}{(1-x^2)(1-2x)}$.

10. $\frac{12x^2-30x+19}{(x-1)(2x-3)(3x-5)}$.

11. $\frac{x^2}{(x^2-1)(x-2)}$.

12. $\frac{x^2}{(x-a)(x-b)(x-c)}$.

13. $\frac{2b^3}{(x-a+b)(x-a-b)(x-a)}$.

14. $\frac{x^2+4x+1}{x^3+2x^2-x-2}$.

15. $\frac{x^2}{(x-1)(x-2)(x-3)}$.

16. $\frac{2x^2-10x+11}{(x-1)(x-2)(x-3)(x-4)}$.

17. $\frac{x^2+1}{(x^2+2)(x^2+3)}$ [Hint: Put $x^2=z$]

11.5. To find the partial fractions when the denominator contains a linear repeated factor.

The following examples illustrate the method.

Ex. 1. Resolve $\frac{2x^2}{(x-1)^3(x+1)}$ into partial fractions.

Here $(x-1)$ is the linear factor repeated three times in the denominator.

$$\text{Let } \frac{2x^2}{(x-1)^3(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$$

$$= \frac{A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3}{(x-1)^3(x+1)}$$

$$\therefore 2x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3$$

In this identity putting $x=-1$, we get $D=-\frac{1}{4}$.

Putting $x=1$, we get $C=1$.

Comparing co-efficients of x^3 and the constant terms we get

$$0 = A + D \text{ and } 0 = A - B + C - D$$

whence $A = -D = \frac{1}{4}$ and $B = A + C - D = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}$

$$\therefore \frac{2x^2}{(x-1)^3(x+1)} = \frac{1}{4(x-1)} + \frac{3}{2(x-1)^2} + \frac{1}{(x-1)^3} - \frac{1}{4(x+1)}$$

Another Method

Put $x-1=y$, so that $x=1+y$.

Hence the fraction $= \frac{2(1+y)^2}{y^3(y+2)} = \frac{2+4y+2y^2}{y^3(2+y)}$

Let us carry out the following process of division :

$$\begin{array}{r} 2+y \overline{) 2+4y+2y^2} \quad (1+\frac{3}{2}y+\frac{1}{2}y^2) \\ \underline{2+y} \\ 3y+2y^2 \\ \underline{3y+\frac{3}{2}y^2} \\ \frac{1}{2}y^2 \\ \underline{\frac{1}{2}y^2+\frac{1}{2}y^3} \\ -\frac{1}{2}y^3 \end{array}$$

$$\therefore \frac{2+4y+2y^2}{2+y} = 1 + \frac{3}{2}y + \frac{1}{4}y^2 - \frac{y^3}{4(2+y)}.$$

Hence the given fraction

$$= \frac{1}{y^3} \left[1 + \frac{3}{2}y + \frac{1}{4}y^2 - \frac{y^3}{4(2+y)} \right].$$

$$= \frac{1}{y^3} + \frac{3}{2} \frac{1}{y^2} + \frac{1}{4} \frac{1}{y} - \frac{1}{4} \frac{1}{(2+y)}$$

$$= \frac{1}{(x-1)^3} + \frac{3}{2} \frac{1}{(x-1)^2} + \frac{1}{4} \frac{1}{(x-1)} - \frac{1}{4} \frac{1}{(x+1)}.$$

It may be noticed that we stop dividing at the stage where the remainder is divisible by y^3 .

Note.—It may be remarked that the first method given above is more useful when the power of the repeated factor is not large. In case it is, then it is advisable to follow the second method.

Ex. 2. Resolve $\frac{x}{(x+1)^3(x-2)^2}$ into partial fractions.

$$\text{Let } \frac{x}{(x+1)^3(x-2)^2}$$

$$= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x-2)} + \frac{E}{(x-2)^2}$$

$$= [A(x+1)^2(x-2)^2 + B(x+1)(x-2)^2 + C(x-2)^2 + D(x-2)(x+1)^3 + E(x+1)^3] \div (x+1)^3(x-2)^2$$

$$\text{Hence } x = A(x+1)^2(x-2)^2 + B(x+1)(x-2)^2 + C(x-2)^2 + D(x-2)(x+1)^3 + E(x+1)^3.$$

Putting $x = -1$ and $x = 2$ successively, we get

$$C = -\frac{1}{9} \text{ and } E = \frac{2}{27}. \text{ Also comparing co-efficients}$$

of x^4 , x^3 and constant term on both sides, we get

$$0 = A + D,$$

$$0 = -2A + B + D + E.$$

$$0 = 4A + 4B + 4C - 2D + E.$$

Solving these equations,

$$A = \frac{1}{27}, B = \frac{1}{27}, D = -\frac{1}{27}.$$

Thus the partial fractions are

$$\frac{1}{27} \frac{1}{x+1} + \frac{1}{27} \frac{1}{(x+1)^2} - \frac{1}{9} \frac{1}{(x+1)^3} - \frac{1}{27} \frac{1}{x-2} + \frac{2}{27} \frac{1}{(x-2)^2}.$$

Note.—The partial fractions corresponding to $(x-a)^r$ in

$$\frac{F(x)}{(x-a)^r \phi(x)} \text{ are } \frac{C_0}{(x-a)^r} + \frac{C_1}{(x-a)^{r-1}} + \dots + \frac{C_{r-1}}{(x-a)};$$

where $C_0, C_1, \dots, C_{r-1}$ are independent of x .

EXERCISE XLVI

Break into Partial Fractions :

1. $\frac{x}{(1-x)^2}$ 2. $\frac{4+5x}{(1-x)^3}$ 3. $\frac{1}{(x^2-1)^2}$ 4. $\frac{5x^2-4x+6}{2(x-2)(x+1)^2}$
5. $\frac{x^2+x+1}{(x-1)^2(x+1)}$ (Q. 1945). 6. $\frac{x^3}{(x-1)^4}$
7. $\frac{x^3+3x^2+5}{(x-1)(x+2)^2}$ 8. $\frac{7x-10}{(3x-4)(x-1)^2}$ 9. $\frac{x}{(x+1)(x^2-1)}$
10. $\frac{2x^2+1}{(x-1)(x+2)^2}$ 11. $\frac{2x^3+1}{(x-1)^3(x+2)}$ 12. $\frac{1}{x^3(x-1)}$
13. $\frac{x-6}{x^3(x+3)}$ 14. $\frac{1}{x^4(x+1)}$ 15. $\frac{x^3}{(x-2)^3(x+1)^2}$
16. $\frac{x^2+1}{(x-2)^3(x-3)}$ 17. $\frac{x^2+1}{(2x-1)^2(x+1)}$ 18. $\frac{1}{(x^2+x)(x^2-1)}$
(P. U. 1950)
19. $\frac{1}{(x^2+1)^2(x^2+3)}$ [Hint. Put $x^2=t$.] 20. $\frac{x^2}{(x^2+a^2)^2(x^2+b^2)}$
21. $\frac{2x^3+x+1}{(x-1)^2(x+3)}$ (Q. 1949).

11 6. To find the partial fractions when the denominator contains a quadratic factor not resolvable into linear factors.

If a factor of the second degree, say ax^2+bx+c , which is not resolvable into real first degree factors, occurs in the denominator, the form for the corresponding partial fraction is

$$\frac{Ax+B}{ax^2+bx+c},$$

and if the factor ax^2+bx+c is repeated any number of times, say, three, the form of the partial fractions corresponding to this would be

$$\frac{Ax+B}{(ax^2+bx+c)^3} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{ax^2+bx+c},$$

where the letters, A, B, C etc., are all independent of x .

The methods of finding out the values of A, B, C etc., will be best illustrated by the solved examples that follow. However, the student will find by experience that he can vary the methods.

Ex. 1. Decompose $\frac{1}{1+x^3}$ into partial fractions.

$$\begin{aligned} 1+x^3 &= (1+x)(1-x+x^2) \\ \text{Let } \frac{1}{1+x^3} &= \frac{A}{x+1} + \frac{Bx+C}{1-x+x^2} \\ &= \frac{A(1-x+x^2) + (Bx+C)(1+x)}{1+x^3} \end{aligned}$$

$$\therefore 1 = A(1-x+x^2) + (Bx+C)(1+x)$$

$$\text{i.e., } 1 = (A+B)x^2 + (B+C-A)x + A+C.$$

In this identity, comparing co-efficients of like powers of x on both sides, we have

$$A+B=0, B+C-A=0, A+C=1.$$

Solving these, we have $A=\frac{1}{3}$, $B=-\frac{1}{3}$, $C=\frac{2}{3}$

$$\therefore \frac{1}{1+x^3} = \frac{1}{3} \cdot \frac{1}{1+x} - \frac{1}{3} \cdot \frac{x-2}{1-x+x^2}.$$

Note.—Observe that A could have been directly found by the mechanical rule of Art. 11.4 or else by putting $x=-1$ in the identity.

Ex. 2. Decompose $\frac{2x+1}{(x^2+2)(x^2+3)}$ into partial fractions.

$$\begin{aligned} \text{Let } \frac{2x+1}{(x^2+2)(x^2+3)} &= \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+3} \\ &= \frac{(Ax+B)(x^2+3) + (Cx+D)(x^2+2)}{(x^2+2)(x^2+3)} \end{aligned}$$

$$\therefore 2x+1=(A+C)x^2+(B+D)x^2+x(3A+2C)+2D+3B.$$

Comparing co-efficients of like powers of x on both sides, we have

$$A+C=0, B+D=0, 3A+2C=2, 2D+3B=1.$$

Solving these we get

$$A=2, B=1, C=-2, D=-1.$$

$$\therefore \frac{2x+1}{(x^2+2)(x^2+3)} = \frac{2x+1}{x^2+2} - \frac{2x+1}{x^2+3} = (2x+1) \left[\frac{1}{x^2+2} - \frac{1}{x^2+3} \right].$$

Note that this is otherwise obvious.

Ex. 3. Decompose $\frac{1}{(x^2+2)^2(x-1)}$ into partial fractions.

$$\begin{aligned} \text{Let } \frac{1}{(x^2+2)^2(x-1)} &= \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+2)^2} + \frac{Dx+E}{(x^2+2)} \\ &= \frac{A(x^2+2)^2 + (Bx+C)(x-1) + (Dx+E)(x^2+2)(x-1)}{(x-1)(x^2+2)^2} \end{aligned}$$

$$\therefore 1 = A(x^2+2)^2 + (Bx+C)(x-1) + (Dx+E)(x^2+2)(x-1).$$

Putting $x=1$, we get $1=9A$, or $A=\frac{1}{9}$.

Comparing co-efficients of x^4 , x^3 , x^2 and x on both sides we have

$$A+D=0, E-D=0, 2D-E+B+4A=0, 2E-2D-B+C=0.$$

Solving these equations, we get

$$B=-\frac{1}{3}, C=-\frac{1}{3}, D=-\frac{1}{9}, E=-\frac{1}{9}.$$

$$\therefore \frac{1}{(x^2+2)^2(x-1)} = \frac{1}{9} \frac{1}{(x-1)} - \frac{1}{3} \frac{x+1}{(x^2+2)^2} - \frac{1}{9} \frac{x+1}{(x^2+2)}.$$

Ex. 4. Resolve $\frac{1}{(x^2+a^2)(x^2+b^2)}$ into partial fractions.

Putting $x^2=z$, we have

$$\begin{aligned} \frac{1}{(z+a^2)(z+b^2)} &= \frac{1}{b^2-a^2} \cdot \frac{1}{(z+a^2)} + \frac{1}{a^2-b^2} \cdot \frac{1}{(z+b^2)} \\ &= \frac{1}{(b^2-a^2)} \cdot \frac{1}{(x^2+a^2)} + \frac{1}{(a^2-b^2)} \cdot \frac{1}{(x^2+b^2)}. \end{aligned}$$

Note.—When all the powers of x are even, the least of them being 2, the work might sometimes be simplified by the substitution $x^2=z$.

EXERCISE XLVII

Decompose into partial fractions :

1. $\frac{7+x}{(1+x)(1+x^2)}$
2. $\frac{1}{1-x^3}$
3. $\frac{x-1}{(x^2+4)(x-2)}$
4. $\frac{2x+1}{(x-1)(x^2+1)}$ (Q. 1947 ; 1950)
5. $\frac{1}{(x+1)(x^3-1)}$
6. $\frac{x-1}{(x+1)(1+x^2)}$
7. $\frac{x}{(x^2+2)(x^2+3)}$
8. $\frac{3x^2+7x+2}{(x+1)(x^2+2x+5)}$
9. $\frac{1}{(x-1)^2(x^2+1)}$
10. $\frac{2x^2+x+2}{(x-2)^2(x^2-x+1)}$
11. $\frac{x}{(x^2+x-2)(x^2-x+2)}$ (P. U. 1950)
12. $\frac{x^3+x^2-x+1}{(x^2+1)(x-1)^2}$ (Q. 1948)
13. $\frac{1}{(x+1)^2(x^2+1)}$ (Q. 1944)
14. $\frac{1}{(x-2)^2(x^2+x+4)}$ (Q. 1946)
15. $\frac{x^3+4x^2+6x+1}{x(x^2+x+1)}$
16. $\frac{x^2+1}{x^4+x^2+1}$
17. $\frac{1}{x^4+1}$
18. $\frac{x^2+1}{(x^2+a^2)(x^2+b^2)}$
19. $\frac{x^3-2}{(x^2+x+2)(x^2+x+1)}$
20. $\frac{x+a}{x^2(x-a)(x^2+a^2)}$
21. $\frac{x}{(1+x)(1+x^2)^2}$
22. $\frac{x^3+x+1}{(x^2+1)^2(x^2+2)}$
23. $\frac{(x^2+1)^2}{(x^3+1)^2}$
24. By taking $x^2+1=(x+i)(x-i)$, put $\frac{x}{(x^2+1)^2}$ into partial fractions.
25. Resolve into partial fractions $\frac{x^5}{x^4-1}$; $\frac{x}{(x^3-1)(x+2)}$ (P. U. 1946)

Applications

11.7. We shall now illustrate some of the applications of partial fractions.

Ex. 1. Sum to n terms the series

$$\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots$$

Here $T_n = \frac{1}{(n+1)(n+2)}$; or putting into partial fractions

$$T_n = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\therefore T_{n-1} = \frac{1}{n} - \frac{1}{n+1}$$

$$T_{n-2} = \frac{1}{n-1} - \frac{1}{n}$$

.....

.....

$$T_2 = \frac{1}{3} - \frac{1}{4}$$

$$T_1 = \frac{1}{2} - \frac{1}{3}$$

$\therefore$ adding up, we have the required sum $= \frac{1}{2} - \frac{1}{n+2}$.

Note.—When n tends to infinity the sum is easily seen to be $\frac{1}{2}$.

***Ex. 2.** Find the co-efficient of x^{2n} in the expansion of

$$\frac{2x+1}{(x-1)(x^2+1)} \quad (x \text{ is positive and } < 1.)$$

Let
$$\frac{2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1},$$

so that $2x+1 = A(x^2+1) + (Bx+C)(x-1).$

Putting $x=1$, we get $A = \frac{3}{2}$.

Also, comparing co-efficients of x^2 and x we have

$$A+B=0, \quad C-B=2.$$

Solving these, we find that $B = -\frac{3}{2}$ and $C = \frac{1}{2}$.

$$\therefore \frac{2x+1}{(x-1)(x^2+1)} = \frac{3}{2} \frac{1}{(x-1)} + \frac{1}{2} \frac{1-3x}{(x^2+1)}$$

$$= -\frac{3}{2}(1-x)^{-1} + \frac{1}{2}(1-3x)(1+x^2)^{-1}.$$

$\therefore$ the required co-efficient of x^{2n} .

$$= \text{Co-efficient of } x^{2n} \text{ in } -\frac{3}{2}(1-x)^{-1} + \frac{1}{2}(1-3x)(1+x^2)^{-1}$$

$$= \text{Co-efficient of } x^{2n} \text{ in } -\frac{3}{2}(1+x+\dots+x^{2n}+\dots)$$

$$+ \frac{1}{2}(1-3x)[1-x^2+x^4-\dots+(-1)^n x^{2n}+\dots]$$

$$= -\frac{3}{2} + \frac{1}{2}(-1)^n.$$

EXERCISE XLVIII

Sum the series to n terms and to infinity :

1. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots$

2. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots$

3. $\frac{1}{(1+x)(1+2x)} + \frac{2}{(1+2x)(1+4x)} + \frac{4}{(1+4x)(1+8x)} + \dots$

Sum to n terms :

4. $\frac{1}{1.2.3} + \frac{4}{2.3.4} + \frac{7}{3.4.5} + \frac{10}{4.5.6} + \dots$

5. $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \dots$

*Find the co-efficient of x^n in the expansion of the following expressions :

6. $\frac{x+2}{(x-1)(x+1)}, x < 1$. 7. $\frac{1}{(2x+1)(x-1)}, |x| < \frac{1}{2}$.

8. Find the co-efficient of x^7 in

$$\frac{1-2x}{3+2x-x^2}, x < 1.$$

9. Find the co-efficient of x^{2n+1} in the expansion of

$$\frac{x+1}{(x-1)(1+x^2)}.$$

10. Resolve $\frac{7-3x+4x^2}{(1+x^2)(1-x)}$ into partial fractions.

Hence expand and show that the co-efficient of all odd powers of x is 4. Find the co-efficients of x^8 and x^{10} .

REVISION QUESTIONS V

Resolve into the partial fractions :

1. $\frac{x+1}{(x+2)(x-1)}$. 2. $\frac{1}{x^2(1+x)}$. 3. $\frac{x^2+3}{(x^2+1)(x^2+2)}$.

4. $\frac{x^5}{x^4-5x^2+4}$. 5. $\frac{2x+1}{(x+2)(x-3)^2}$. 6. $\frac{1}{x^4+x^2+1}$.

7. $\frac{x^3}{(x^2+a^2)(x^2+b^2)}$ 8. $\frac{1}{1-x^4}$ 9. $\frac{1}{x^4(1+x^2)}$
 10. $\frac{x^2-1}{x^4+x^2+1}$ 11. $\frac{1}{x(x^2+1)^3}$ 12. $\frac{x^3+x+1}{(x^2+1)^2(x+2)}$
 13. (i) $\frac{x-1}{(x^2+4)(x-2)}$ (ii) $\frac{x+1}{(x^2-4)(x+2)}$ (P. U. 1948)
 14. (i) $\frac{x+1}{(x-1)^3(x+2)}$ (ii) $\frac{x^2+2x+2}{(x^2-1)(x^2+4)}$ (P. U. 1947)

Sum up :

15. $\frac{1}{(1+x)(2+x)} + \frac{1}{(2+x)(3+x)} + \frac{1}{(3+x)(4+x)} + \dots$ to n terms.
 16. $\frac{x+2}{(1+x)(3+2x)} + \frac{2x+2}{(3+2x)(5+4x)} + \frac{4x+2}{(5+4x)(7+8x)} + \dots$ to infinity.

17. Find the co-efficient of x^r in the expansion of $\frac{1+3x+4x^2}{(1-x)^3}$. (Q. 1948)

18. In the expansion of $\frac{3x-8}{4-4x+x^2}$ in the ascending powers of x , find the co-efficient of (i) x^4 . (ii) x^r .

19. Find the co-efficient of x^{2n} in the expansion of $\frac{x^2-x+2}{(1+x^2)(1-x)}$ in ascending powers of x .

20. Resolve into partial fractions $\frac{5x+3}{(x+1)^2(2x+1)}$.

Hence find the co-efficient of x^n in its expansion in ascending powers of x and state the range of values of x for which the expansion is valid.

21. Express $\frac{1}{(1-x)(1-2x)(1-3x)}$ in the form of partial fractions and hence show that the co-efficients of x^n in the given fraction is $\frac{1}{2} [3^{n+2} - 2^{n+3} + 1]$. (P. U.)

22. Show that $\frac{x^{n+1}}{(x-a_1)(x-a_2)\dots(x-a_n)} = x + a_1 + a_2 + \dots + a_n + \sum \frac{a_1^{n+1}}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)} \cdot \frac{1}{x-a_1}$.

HIGHER SECONDARY EXAMINATION DELHI

(Selected Questions from 1946-52 Papers)

1. Solve :

$$(i) \quad \frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}$$

$$(ii) \quad \sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = 2\frac{1}{x}$$

2. If α, β be the roots of $x^2 - px + q = 0$ show that $\frac{1}{a\alpha + b}$ and $\frac{1}{a\beta + b}$ will be roots of

$$(a^2q + abp + b^2)x^2 - (ap + 2b)x + 1 = 0.$$

3. Employ the Binomial Theorem to find $(999)^{\frac{1}{3}}$.

4. Solve $\frac{57}{x+y} + \frac{6}{x-y} = 5.$

$$\frac{38}{x+y} + \frac{21}{x-y} = 9.$$

5. If $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have one root common and the second equation have equal roots show that

$$b + q = \frac{a + p}{2}.$$

6. If S_n denote the sum of n terms of an A.P., whose first term is a then show that common difference is

$$d = S_n - 2S_{n-1} + S_{n-2}.$$

7. If a, b, c be in A.P., b, c, d in G.P., c, d, e in H.P., then prove that $c^2 = ae$.

8. Show that $1 + 2x + 3x^2 + 4x^3 + \dots$ to n terms

$$= \frac{1-x^{n+1}}{(1-x)^2} - \frac{(n+1)x^{n+1}}{1-x}.$$

9. Find for what values of m will the equation $x^2 - 2x(1+3m) + 7(3+2m) = 0$ have equal roots.

10. Solve the equations :

$$(i) \quad (x+2)(x+3)(x+5)(x+6) = 504,$$

$$(ii) \quad x^2 + 2\sqrt{x^2 + 2x + 3} = 12 - 2x.$$

11. (i) If α, β be roots of $x^2 - ax + b = 0$, form the equation whose roots are $\frac{1}{\alpha + \beta}$ and $\frac{1}{\alpha\beta}$.

(ii) If α and β are the roots of $3x^2 + 2x + 1 = 0$, find the equation whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$.

12. Show that if a, b, c are in A. P., and b, c, a in G. P., then c, a, b will be in H. P.

13. (a) Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

(b) Write down the 5th term of $\left(\frac{4x}{4} - \frac{5}{2x}\right)^8$ and show that it is independent of x .

(c) Find the two middle terms in the expansion of

$$\left(x - \frac{1}{x}\right)^{11}$$

14. If P be the sum of odd terms and Q the sum of even terms in the expansion of $(x+a)^n$, prove that $P^2 - Q^2 = (x^2 - a^2)^n$
 $4PQ = (x+a)^{2n} - (x-a)^{2n}$.

15. A man rides one-third of the distance from A to B at the rate of a miles an hour and the remainder at the rate of $2b$ miles an hour. If he had travelled at a uniform rate of $3c$ miles per hour, he could have ridden from A to B and back again in the same time. Prove that a, c, b are in H. P.

16. Solve (i) $16x(x+1)(x+2)(x+3) = 9$.

(ii) Find the value of

$$2x^3 - 9x^2 - 10x + 13 \text{ when } x = 3 + \sqrt{5}.$$

17. If $b+c, c+a, a+b$ are in H. P. prove that a^2, b^2, c^2 are in A. P.

18. Show that 5th term in the expansion of $(1-2x^3)^{11}$ is $\frac{1155}{8}x^{12}$.

19. Solve $\frac{2}{x-2} + \frac{3}{x-3} = \frac{10}{x-5}$.

20. If roots of the equation $ax^2 + bx + c = 0$ be in the ratio $p : q$ then $(p+q)^2 = b^2/pq$.

21. If a^2, b^2, c^2 be in A. P.,

show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A. P.

22. Find three numbers in G. P. whose product is 343 and sum 35.

23. (a) Find the term independent of x in

$$\left(3x^2 - \frac{9}{3x^3}\right)^{10}$$

(b) If in the expansion of $(1+x)^n$ the fifth term be four times the 4th and 4th term be six times the third term; find the values of n and x .

24. Four persons are chosen out of 10, in how many ways can this be done and how often would a person be chosen?

25. Find the term independent of x in the expansion of $\left(x - \frac{1}{x^2}\right)^{3n}$.

26. Solve:

$$(i) x^{\frac{2}{3}} + 12 = 7x^{\frac{1}{3}}. \quad (ii) x^2 + y^2 = 229, x - y = 13.$$

27. If α, β be the roots of $px^2 - qx + r = 0$ form the equation whose roots are $\alpha^2 + \beta^2$ and $2\alpha\beta$.

28. Write down 7th term of

$$\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$$

29. Solve:

$$(i) 5\sqrt{\frac{3}{x}} + 7\sqrt{\frac{x}{3}} = 22\frac{2}{3}.$$

$$(ii) x + \frac{1}{y} = 1, y + \frac{1}{x} = 4.$$

30. Find the term independent of x in $\left(x^2 - \frac{1}{x}\right)^{2n}$.

31. If x be nearly 1 show that

$$\frac{mx^m - nx^n}{m-n} = x^{m+n}.$$

32. If a, b, c be in H. P. show that $a^3 + c^3 > 2b^3$.

33. Resolve into partial fractions:

$$\frac{x^2}{(x-1)^3(x+1)}.$$

PUNJAB UNIVERSITY PAPERS

1949

1. (a) Solve the equation $ax^2+bx+c=0$, and discuss the nature of its roots.
 (b) Show that the roots of the equation $(x-a)(x-b)=h^2$ are always real.

2. (a) Find the sum of the first n terms of a G. P., the common ratio being given.

(b) Sum the series

(i) $5+55+555+\dots$ to n terms.

(ii) $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\dots$ to infinity.

3. (a) Prove that

$${}^nC_r = {}^nC_{n-r} \text{ and evaluate } {}^{50}C_{47}.$$

(b) If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^rC_5$.

4. (a) Solve the equation

$$12+9\sqrt{(x-1)(3x+2)}=3x^2-x.$$

(b) If $b+c, c+a, a+b$ are in H. P., show that

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in A. P.}$$

Or,

(b) Resolve

$$\frac{1}{(x+1)^2(x^2+1)}$$

into partial fractions.

5. (a) In the Intermediate Examination a candidate has to pass in each of four subjects. In how many ways can he fail?

(b) Find the sum of the series

$$\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \dots$$

1950

1. (a) Find the condition that $ax^2+bx+c=0$ and $a^1x^2+b^1x+c^1=0$ may have a common root.

(b) Solve (i) $\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$.

(ii) $x-y=1, x^3-y^3=19$.

2. (a) Show that the sum of the cubes of the first n natural numbers is equal to the square of the sum of these numbers.

(b) Find the sum of the infinite series

$$1 + (1+b)r + (1+b+b^2)r^2 + (1+b+b^2+b^3)r^3 + \dots$$

r and b are proper fractions;

3. (a) Without using any formula, prove that

$${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}.$$

Hence or otherwise find the value of ${}^nC_r$.

(b) 7 candidates are to take examination, 2 in Mathematics and the remaining in different subjects. In how many ways can they be seated in a row so that the two examinees in Mathematics may not sit together?

4. Resolve into partial fractions :

$$(i) \frac{x}{(x^2+x-2)(x^2-x+2)}.$$

$$(ii) \frac{1}{(x^2+x)(x^2-1)}.$$

5. (a) If $p=q$ nearly and $n > 1$ show that

$$\frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} = \left(\frac{p}{q}\right)^{\frac{1}{n}}.$$

Apply this rule to find the seventh root of $\frac{131}{138}$.

(b) If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, find by differentiation or otherwise, the value of

$$C_1 + 2C_2 + 3C_3 + \dots + nC_n.$$

Or,

(b) Euler (a mathematician) says it would be difficult to find x so as to make $p^4 - 2p^2x^2 + x^4 + x$ a square, show that it is not difficult.

1951

1. Find the sum of n term of the series :

$$a + ar + ar^2 + \dots + ar^{n-1}.$$

Obtain the sum of an infinite number of terms when $r < 1$.

Prove that $\cdot 45 = \frac{5}{11}$.

Find the sum of p terms of the series 2, 8, 26, 80.....

2. Prove that a quadratic equation cannot have more than two roots.

Solve : (i) $\sqrt{x+5} + \sqrt{x+12} = \sqrt{2x+41}$

(ii) $x^2 - 7xy + 12y^2 = 0$

$x^2 + 5xy - 8y^2 = 34$

3. Find the number of permutations of n dissimilar things taken r at a time. In how many of them will three particular things occur?

Find how many words can be formed from the letters of the word 'Oriental' so that (a) and (e) always occupy odd places.

4. Prove Binomial theorem for a positive integral index.

If three successive coefficients in the expansion $(1+x)^n$ are 45, 120 and 210, find n .

5. Resolve into partial fractions :

(i) $\frac{1}{(x^2+5x+6)(x^2+1)}$

(ii) $\frac{x}{(x-1)^2(x-2)}$

1952

1. (a) Discuss the nature of the roots of the equation $ax^2+bx+c=0$. Form the equation whose roots are the squares of the roots of the above equation.

(b) Solve the equations :—

(i) $\sqrt{\frac{x}{x-1}} + \sqrt{\frac{1-x}{x}} = 2\frac{1}{6}$

(ii) $\frac{x}{3} + \frac{y}{5} = 3$

$\frac{3}{x} + \frac{5}{y} = 1\frac{1}{2}$

2. Find the nc_r without assuming the formule for nc_r , and deduce that $nc_r = nc_{n-r}$.

How many triangles can be formed by joining 12 points 7 of which are in the same straight line?

3. Write down the expansion of $(1+x)^n$ when n is not a positive integer. Give the number of terms in the expansion and state the limits in the value of x .

Find the cube root of 126 correct to seven decimal places.

4. Resolve into partial fractions :

(i) $\frac{x}{x^2+1}$ (ii) $\frac{x^2+1}{(x-2)^2(x+3)}$

6. If m be the arithmetic mean of n consecutive integers, show that the sum of their cubes is

$$mn \left[m^2 + \frac{n^2 - 1}{4} \right].$$

DELHI UNIVERSITY QUALIFYING EXAMINATION PAPERS

1949

1. (a) Solve the equation

$$\sqrt{x^2 - 16} - (x - 4) = \sqrt{x^2 + 5x + 4},$$
 (b) Break up into partial fractions

$$\frac{2x^2 + x + 1}{(x - 1)^2(x + 3)}.$$
 (c) Find the square root of $1 + 2\sqrt{6i}$ where i stands for the square root of -1 .
2. (a) Show that a quadratic equation cannot have more than two roots.
 (b) If α and β are the roots of $x^2 - px + q = 0$, find the equation whose roots are α^2 and β^2 .
 (c) Find the condition that

$$ax^2 + bx + c = 0$$
 and
$$a'x^2 + b'x + c' = 0,$$
 may have a common root.
3. Sum the series :
 (a) $1^2 + 2^2 + 3^2 + \dots + n^2.$
 (b) $8 + 88 + 888 + 8888 \dots$ to n terms.
 (c) $1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1}.$
4. Find the number of combinations of n different things taken r at a time.
 A council consists of eleven members of which five are permanent and the remaining six are temporary. A resolution is declared to be passed by the council if it receives at least seven votes including all the five of the permanent members. Find the number of ways in which the resolution can be passed.
 What will be the number of ways, if
 (i) for passing a resolution, only a simple majority was required,
 (ii) any seven or more votes could secure the passing of the resolution?

5. (a) Find the value of the square root of 999 correct to three places of decimals.
 (b) Write down the general term and the co-efficients of x^{12} and x^{-6} in the expansion of $\left(x^2 + \frac{1}{x^7}\right)^{15}$.

1950

1. Solve the equations :
 (i) $(x+2)(x+3)(x+5)(x+6)=504$.
 (ii) $\sqrt{2x-5}-\sqrt{4x-9}=\sqrt{4x+7}$.
 (iii) $x+y=16, x^3+y^3=1072$.
2. (a) Examine the nature of the roots of the quadratic equation $ax^2+bx+c=0$.
 (b) If α, β be the roots of $2x^2-3x-5=0$, form the equation whose roots are $\frac{\alpha}{\alpha^2+\beta^2}, \frac{\beta}{\alpha^2+\beta^2}$.
 (c) Reduce $\frac{a+ib}{a-ib}$ to the form $x+iy$.
3. (a) Find the sum of n terms of a G. P.
 (b) If $a^p=b^q=c^r$ and a, b, c be in G. P. prove that p, q, r are in H. P.
 (c) Find the sum of n terms of the series
 $1^2+3^2+5^2+7^2+\dots\dots\dots$
4. Prove that ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$.
 Find the number of combinations of the letters of the word 'College' taken four together.
5. (a) Find the term independent of x in
 $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.
 (b) Extract the fifth root of 244 correct to three places of decimals.
 (c) Resolve $\frac{2x+1}{(x-1)(x^2+1)}$ into partial fractions.

1951

1. Solve the equations :
 (i) $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = 2\frac{1}{6}$.

- (ii) $\sqrt{5x^2-6x+8}-\sqrt{5x^2-6x-7}=1.$
 (iii) $x^4-2y^4=1, x^2-y^2=1.$
2. (a) Prove that a quadratic equation cannot have more than two roots.
 (b) If the roots of $x^2-px+q=0$ differ by unity, prove that $p^2=1+4q.$
 (c) Find the square root of $3-4\sqrt{-7}.$
3. (a) Find the sum of n terms of an A. P., the first term and the common difference being given.
 (b) If a, b, c be in A. P., b, c, d , in G. P., c, d, e in H. P., prove that $c^2=ae.$
 (c) Sum the series $1+3x+6x^2+10x^3+\dots$ to infinity x being $<1.$
4. Find the number of permutations of n different things taken r at a time.

In how many ways can 5 persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements?

5. (a) Find the two middle terms of $\left(3a-\frac{a^3}{6}\right)^9.$
 (b) If x be nearly equal to 1, show that

$$\frac{ax^b-bx^a}{x^b-x^a}=\frac{1}{1-x}$$
 nearly.
 (c) Resolve $\frac{x^2}{(x-1)^3(x+1)}$ into partial fractions.

1952

1. (a) Find the condition that the quadratic expression ax^2+bx+c will have a positive value for any real value of $x.$
 (b) If the equation $x^2-15-m(2x-8)=0$ has equal roots, find the value of $m.$
 (c) If α, β are the roots of the equation $x^2-px+q=0,$ find the value of $\alpha^3+\beta^3.$
2. Solve the equations:
- (i) $\sqrt{2x^2-7x+1}-\sqrt{2x^2-9x+4}=1$
 (ii) $(x+9)(x-3)(x-7)(x+5)=385$
 (iii) $x^2-y^2=20$ and $x+y=10.$

3. (a) Find the sum of the first n terms of a G. P., the first term and the common ratio being given.
 (b) Sum the series $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \dots$ to infinity.
 (c) Find the sum of the squares of the first n odd numbers.
4. (a) Find the number of combinations of n different things taken r at a time.
 (b) Prove that

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}.$$

 (c) A regular polygon has 23 vertices and, consequently 23 sides. How many additional lines need to be drawn so that every pair of vertices will be connected?
5. (a) State and prove the binomial theorem for a positive integral index.
 (b) If P be the sum of the odd terms and Q the sum of the even terms in the expansion of $(a+b)^n$, where n is a positive integer, prove that

$$P^2 - Q^2 = (a^2 - b^2)^n.$$

 (c) Resolve $\frac{1}{x^3-1}$ into partial fractions.

DELHI UNIVERSITY
ENTRANCE SCHOLARSHIP EXAMINATION PAPERS

1946

1. (a) Solve the equations :
 (i) $\sqrt{x^2+2x-3} + \sqrt{x^2-x} = 5\sqrt{x-1}.$
 (ii) $x^2 - y^2 = 7, x - y = 1.$
 (b) If $x^2 - 9x + 20$ and $x^2 - 13x + a$ have a common factor, find a .
2. (a) If a, b, c , be in G.P. and p is the Arithmetic mean between a and b , and q is the Arithmetic mean between b and c . Prove that b is the Harmonic mean between p and q .
 (b) Sum the series

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots \text{to } n \text{ terms.}$$

2. (a) Prove that ${}^{n+1}P_r = {}^nP_r + r \cdot {}^nP_{r-1}$.

(b) There are 16 persons including 5 ladies. In how many ways can a committee of 4 be chosen so as to include (i) one lady (ii) at least one lady.

4. (a) Resolve into partial fractions

$$\frac{x^4}{x^3-1}$$

(b) Use the Binomial Theorem to evaluate $\sqrt{101}$ to four places of decimals.

1947

1. Solve the equations :—

$$(i) \quad x^2 - \frac{2}{3x} = 1\frac{4}{9}.$$

$$(ii) \quad \sqrt{3x^2-5x-6} - \sqrt{3x-5x+3} = 9.$$

2. Construct a quadratic equation with rational co-efficients one of whose roots is $p + \sqrt{q}$.

If r be one root of $4x^2 + 2x - 1 = 0$, prove that the other root is $4\alpha^3 - 3\alpha$.

3. Prove that three different quantities cannot be in H.P. and also in A.P. unless they are also in G.P.

α, β, γ are the Geometric means between ca and ab ; ab and bc ; bc and ca , respectively. Prove that if a, b, c are in A.P. $\alpha^2, \beta^2, \gamma^2$, are in A.P. and $\beta + \gamma, \gamma + \alpha$ and $\alpha + \beta$ are in H.P.

4. Sum the series :

$$(i) \quad 4 + 8 + 16 + \dots \text{to infinity.}$$

$$(ii) \quad 3 + 6 + 11 + 20 + 37 + \dots \text{to } n \text{ terms.}$$

5. (a) At an election where every voter may vote for any number of candidates not greater than the members to be elected, there are four candidates and three members to be chosen. In how many ways a man can vote ?

(b) If x^r occurs in the expansion of

$$(x^4 + x^{\frac{1}{2}})^{2n},$$

find its co-efficient.

1948

1. Solve the equations :—

$$(i) \quad \sqrt{2x-5} - \sqrt{4x-9} = \sqrt{4x+7}.$$

$$(ii) \quad x^4 - 2y^4 = 1, \quad x^2 - y^2 = 1.$$

2. Find the condition that $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ may have both roots common.

If quadratic equations be formed whose roots differ from those of $x^2+px+q=0$ by the same quantity, prove that the discriminants are the same.

3. Find the n th term and the sum to n terms of the series

$$6+13+24+39+58+\dots$$

If a, b, c and d be in H.P., prove that

$$a+d > b+c.$$

4. Prove that ${}^{n+1}P_r = {}^nP_r + r \cdot {}^nP_{r-1}$.

In how many ways can 6 different rings be worn on the four fingers of one hand?

5. (a) If x be nearly equal to 1, show that

$$\frac{mx^m - nx^n}{m-n} = x^{m+n} \text{ nearly.}$$

(b) Resolve :

$$\frac{x}{x^4+x^3+1} \text{ into partial fractions.}$$

ANSWERS

EXERCISE 1, Page 4

1. (i) $\frac{1}{3}, \frac{1}{2}$ (ii) $-\frac{2}{3}, \frac{1}{2}$ (iii) 0, 1, -2. 2. $\frac{1}{2}p, \frac{1}{4}q$.
 3. ± 1 . 4. $\pm(a+b)$ 5. 5. 6. ± 2 . 7. 5.
 8. ± 5 . 9. $\frac{-7 \pm \sqrt{-11}}{6}$. 10. $\frac{-2 \pm \sqrt{58}}{3}$.

EXERCISE II-A, Page 6

1. $\frac{2}{3}, -\frac{7}{2}$. 2. $\frac{-5 \pm \sqrt{-47}}{12}$. 3. $b \pm 1$. 4. $\pm \sqrt{3}$.
 5. $\pm 2a$. 6. $a-2b, -(a+b)$. 7. 3, -1, -9. 8. 0, 2, 3.
 9. $a-2b$. 10. $2a-b, a-2b$ 11. a^2, a^2-1 . 12. $b-2a, -a-b$.
 13. $a+b, 2b$. 14. 0, $-\frac{a^2+b^2}{2ab}$. 15. $a, \frac{-3a}{2}$.
 16. $\frac{9 \pm \sqrt{19}}{2}$. 17. 0, $\frac{1}{3}$. 18. $\frac{2}{a+b}, \frac{a+b}{ab}$.
 19. $\frac{a+b}{a}, \frac{a+b}{b}$. 20. $b, \frac{b}{2}$. 21. $\pm a, \pm a$.
 22. $\frac{-11 \pm \sqrt{13}}{6}b$.

EXERCISE II-B, Page 9

1. $\pm \sqrt{1 - \frac{(a-2)^2}{27a}}$. 2. $\frac{\pm 5a}{3\sqrt{3}}$. 3. 0, $\pm \frac{l}{m}$.
 4. 4. 5. 0. 6. $-\frac{1}{2}, -\frac{1}{2}$. 7. $\pm \sqrt{\frac{3}{5}}$.
 8. 4, $-\frac{1}{2}$. 9. -1, $\frac{1}{2}$. 10. $\frac{2}{3}, \frac{4}{3}$; (ii) $\frac{5}{3}, \frac{8}{3}$. 11. 4, 8.

EXERCISE III, Page 11

1. 10, 20. 2. 7, 8, 9, 10, 11, 12. 3. 5 cms.
 4. 45, 55 m.p.h. 5. 15 ft. 4 in., 12 ft.

EXERCISE IV, Page 12

1. $\pm \sqrt{3}, \pm \sqrt{2}$. 2. $\pm 2, \pm \sqrt{2}, \pm \sqrt{-2}, \pm \sqrt{-4}$.

3. $1, 2, \frac{-1 \pm \sqrt{-3}}{2}, -1 \pm \sqrt{-3}.$ 4. $1, -8.$
 5. $(-2a)^n, (3a)^n.$ 6. $2, 3.$ 7. $\pm \frac{1}{2}.$ 8. $2, 0.$ 9. $\frac{4}{13}, \frac{9}{13}.$
 10. $2 \pm \sqrt{5}, \frac{1 \pm \sqrt{5}}{2}.$ 11. (i) $\frac{3 \pm \sqrt{13}}{2}, \frac{3}{2} \pm \frac{3}{4} \sqrt{5}.$
 (ii) $-1 \pm \sqrt{7}, -1 \pm \sqrt{23}$ (Extraneous).
 12. $0, -2, 4, \frac{2}{3}.$ 13. $1, \frac{-11 \pm \sqrt{105}}{4}.$ 14. $\pm \sqrt{\frac{5}{2}}, \pm \sqrt{\frac{10}{7}}.$
 15. $-1 \pm \frac{3}{4} \sqrt{2}$ 16. $a - 2b, b - 2a.$

EXERCISE V, Page 14

1. $\frac{-5 \pm \sqrt{5}}{2}.$ 2. $2, -1, \frac{1 \pm \sqrt{17}}{2}.$
 3. $9, -7, 1 \pm \sqrt{-24}.$ 4. $a, -9a, (-4 \pm \sqrt{-15})a.$
 5. $0, \frac{1}{3}, \frac{2 \pm \sqrt{13}}{12}.$ 6. $-4, 2, -1 \pm \sqrt{-8}.$
 7. $1, -9, -4 \pm \sqrt{-20}.$ 8. $-\frac{3}{2}, -\frac{3}{2}, \frac{-3 \pm \sqrt{10}}{2}.$

EXERCISE VI, Page 16

1. $1, 2, \frac{1}{2}.$ 2. $-1, \frac{-5 \pm \sqrt{21}}{2}.$
 3. $2, \frac{1}{2}, -1.$ 4. $\pm 1, \frac{-3 \pm \sqrt{5}}{2}.$ 5. $-1, -1, -2, -\frac{1}{2}.$
 6. $-1, -2, -\frac{1}{2}, \frac{3 \pm \sqrt{5}}{2}.$ 7. $2, \frac{1}{2}, -3, -\frac{1}{3}.$

EXERCISE VII, Page 16

1. $1, -5, -6.$ 2. $-2, 7.$ 3. $9, -1, \frac{1}{8}.$
 4. $-\frac{a}{3}, a, \frac{a}{2}.$ 5. $4, 5.$ 6. $2.$ 7. $4, 5.$

EXERCISE VIII, Page 17

1. $1, 2, \frac{1}{2}.$ 2. $2, \frac{3 \pm \sqrt{29}}{2}.$ 3. $-1, -1, 1.$

4. $\pm 1, -\frac{3}{2}$.

5. $2, \frac{16}{17}$.

6. $\frac{-7 \pm \sqrt{17}}{8}$.

7. $a, 0$.

8. $4, -\frac{14}{5}$.

EXERCISE IX, Page 19

1. $x=2, y=1; x=-\frac{2}{7}, y=-\frac{17}{7}$.

2. $x=3, y=1; x=\frac{1}{2}, y=-\frac{1}{4}$.

3. $x=1, y=2; x=-3, y=-2$.

4. $x=2a, y=2b; x=0, y=0$. 5. $x=3, y=4; x=2\frac{5}{8}, y=2\frac{5}{8}$.

6. $x=3, y=1; x=\frac{5}{11}, y=-\frac{10}{11}$.

7. $x=1, y=3; x=15\frac{1}{4}, y=-1\frac{3}{4}$.

8. $x=a, y=b; x=\frac{a-b}{2}, y=\frac{b-a}{2}$.

9. $x=3, y=2; x=-3, y=-2$.

10. $x=11, y=8$.

11. $y=0, \pm\sqrt{2}; x=\pm 1, \pm\sqrt{3}$.

12. $x = \frac{b \pm \sqrt{b^2 + a^2 + ab^2}}{(a+b)}, y = \frac{a \mp \sqrt{b^2 + a^2 + ba^2}}{a+b}$.

13. $x=\frac{1}{2}, -\frac{1}{4}; y=\frac{1}{4}, -\frac{1}{2}$.

EXERCISE X, Page 21

1. $x=2, y=3; x=3, y=2; x=-2, y=-3; x=-3, y=-2$.

2. $x=\frac{1}{2}, y=\frac{3}{2}; x=\frac{3}{2}, y=\frac{1}{2}; x=-\frac{1}{2}, y=-\frac{3}{2}; x=-\frac{3}{2}, y=-\frac{1}{2}$.

3. (a) $x = \frac{5 \pm \sqrt{15}}{2}, y = \frac{5 \mp \sqrt{15}}{2}$.

(b) $x=11, y=8; x=-8, y=-11$.

4. $(4, 3); (-3, -4)$

5. $x=3, y=4$

6. $x=3, y=2$

$x=4, y=3$

$x=-2, y=-3$.

7. $x=7, y=5; x=-5, y=-7$

8. $x=7, y=9; x=9, y=7$.

9. $x=2, y=1; x=1, y=2$

10. $x=9, y=3; x=3, y=9$.

11. $x=2, y=3; x=3, y=2$

12. $x=5, y=1; x=-1, y=-5$.

13. $x=b, y=a; x=\frac{a^2}{b}, y=\frac{b^2}{a}$

14. $x=\frac{1}{2}, y=\frac{1}{2}; x=-\frac{1}{2}, y=-\frac{1}{2}$.

15. $x=a, y=0; x=0, y=-b$.

16. $x=9, y=1; x=1, y=9$.

17. $x=6, y=10; x=4, y=15$.

EXERCISE XI, Page 22

1. $x = \frac{11\sqrt{2}}{2}, y = \frac{3\sqrt{2}}{2}; x = -\frac{11\sqrt{2}}{2}, y = -\frac{3\sqrt{2}}{2};$

$x=7, y=4; x=-7, y=-4$.

$$2. \quad x=3, y=5; x=5, y=3; x=-3, y=-5; x=-5, y=-3$$

$$3. \quad x=\frac{1}{\sqrt{2}}, y=\frac{1}{\sqrt{2}}; x=-\frac{1}{\sqrt{2}}, y=-\frac{1}{\sqrt{2}};$$

$$x=\frac{2}{\sqrt{5}}, y=\frac{1}{\sqrt{5}}; x=-\frac{2}{\sqrt{5}}, y=-\frac{1}{\sqrt{5}}.$$

$$4. \quad x=1, y=2; x=\sqrt[3]{\frac{1}{2}}, y=3\sqrt[3]{\frac{1}{2}}.$$

$$5. \quad x=\frac{1}{2}[b \pm \sqrt{2a^2 - b^2}], y=\frac{1}{2}[b \mp \sqrt{2a^2 - b^2}].$$

$$6. \quad x=2, y=5; x=-2, y=-5; x=\frac{7}{\sqrt{3}}, y=-\frac{1}{\sqrt{3}};$$

$$x=-\frac{7}{\sqrt{3}}, y=\frac{1}{\sqrt{3}}.$$

$$7. \quad x=\frac{1}{8}, y=\frac{1}{8}; x=\frac{1}{8}, y=\frac{1}{2}. \quad 8. \quad 18 \text{ ft.}, 16\frac{1}{2} \text{ ft.}$$

EXERCISE XII, Page 24

$$1. \quad x=3, y=5, z=4; x=-3, y=-5, z=-4.$$

$$2. \quad x=1, y=2, z=-3; x=-\frac{1}{8}, y=\frac{1}{8}, z=-\frac{9}{8}.$$

$$3. \quad x=\pm 5, y=\mp 3, z=\pm 8.$$

$$4. \quad x=5, y=9, z=11; x=-5, y=-9, z=-11.$$

$$5. \quad x=\pm \frac{a^3}{bc}, y=\pm \frac{b^3}{ac}, z=\pm \frac{c^3}{ab}.$$

$$6. \quad x=\pm 4, y=\pm 1, z=\pm 7.$$

$$7. \quad x=3, y=4, z=5; x=-3, y=-4, z=-5.$$

$$8. \quad x=6, y=9, z=1; x=6, y=-1, z=-9.$$

$$9. \quad z=6, y=3, x=5; z=-8, y=-5, x=-7.$$

$$10. \quad x=2\frac{2}{7}, y=1\frac{4}{7}, z=-12.$$

$$11. \quad x=4, y=6, z=2, x=8\frac{1}{4}, y=9\frac{3}{4}, z=-6.$$

$$12. \quad x = \frac{b+c-a}{\pm 2\sqrt{a^2+b^2+c^2}}, y = \frac{c+a-b}{\pm 2\sqrt{a^2+b^2+c^2}},$$

$$z = \frac{a+b-c}{\pm 2\sqrt{a^2+b^2+c^2}}.$$

$$18. \quad x=9, y=-9, z=6; 198.$$

EXERCISE XIII, Page 25

$$1. \quad \frac{1}{2} \text{ m p.h.}, 5 \text{ m.p.h.}$$

$$2. \quad 10, 30.$$

$$8. \quad 45, 54.$$

$$4. \quad 30.$$

EXERCISE XV—A Page 29

1. (i) $p^3 - 3pq$, (ii) $\frac{p^3 - 3pq}{q^3}$,
 (iii) $\frac{p^3 - 2pq + q}{p}$, (iv) $p^4 - 4p^2q + 2q^2$.
 2. 6, -2. 3. $\frac{11}{11}$.
 4. (i) $\frac{b^4 - 4acb^2 + 2a^2c^2}{a^2c^2}$, (ii) $\frac{b^2(b^2 - 4ac)}{a^2c^2}$.
 5. $a^2r^2 + ab(q^2 - 2r) + b^2$. 6. (i) $\frac{q}{\sqrt{pr}}$, (ii) $\frac{2bp + aq}{b^2p + abq + a^2r}$.
 7. (i) $\frac{1}{4q - p^2}$, (ii) $qp^2 - 2q^2$. 8. $2q + 2q' + pp'$.
 9. -1, 6. 10. $\frac{4}{3}, -\frac{1}{3}, -\frac{4}{3}$. 12. $b^2 = 9c^2 + 4c$.

EXERCISE XV—B Page 34

1. (i) $x^2 - 22x + 120 = 0$. (ii) $6x^2 + x - 1 = 0$.
 (iii) $x^2 - 2ax + a^2 - b^2 = 0$. (iv) $x^2 - 4x + 1 = 0$.
 (v) $x^2 - 4x + 7 = 0$. 2. $1 - \sqrt{2}, -1$.
 3. $qx^2 - (p^2 - 2q)x + q = 0$. 4. (a) $x^2 - x(2c + b^2) + c^2 = 0$.
 (b) $x^2 - 2x + 3 = 0$.
 5. $p^2x^2 - q^2px + 2r(q^2 - 2pr) = 0$. 6. $(29x)^2 - 174x - 40 = 0$.
 7. $x^2 - 2x + 1 = 0$. 8. (i) $x^2 - p^2x + p^2q = 0$.
 (ii) $pqx^2 + (q - p)x - 1 = 0$.
 9. $a^4x^2 - 2a^2(b^2 - 2ac)x + b^2(b^2 - 4ac) = 0$.
 11. (a) $\frac{7}{8}$. (b) $2, -\frac{1}{8}$. 13. $ac(a + c - 3b) + b^2 = 0$.
 15. 3. 16. $7x^2 - x + 7 = 0$.
 18. (i) $a = c$. (ii) c/a is positive and b/a is negative i.e., c and a have the same sign opposite to that of b . (iii) c/a and b/a are positive, i.e., a, b and c have the same sign. (iv) c/a is negative, i.e., c and a have opposite signs. (v) $b = 0$.
 19. (i) $c = 0$, (ii) $c = 0, b = 0$. (iii) $a = 0$.
 (iv) $a = 0, b = 0$, (v) $a = c = 0$. 20. $x^2 - (a^2 - 2b)x + b^2 = 0$.
 21. $x^2 + \sqrt{\frac{b}{a}}x + 1 = 0$.

EXERCISE XVI, Page 37

3. $p = -5$; $2x - 3$.

EXERCISE XVII, Page 41

1. (i) $-a\sqrt{xy}$. (ii) $a^2 + x$. 2. (b) $x = \frac{a^2 - b^2}{a^2 + b^2}$, $y = \frac{2ab}{a^2 + b^2}$.

8. 4. 9. 2.

EXERCISE XVIII—A, Page 43

1. (i) real, distinct and rational, (ii) real, distinct and irrational, (iii) real and equal, (iv) real.

2. $-\frac{9}{16}$. 6. $\frac{1 \pm 4\sqrt{2}}{2}$. 7. $-\frac{13}{4}$, 3. 8. 0, 3. 9. $mc = a$.

EXERCISE XVIII—B, Page 45

4. $x > 4$.

REVISION QUESTIONS I Page 47

1. (i) 4, $\frac{10}{11}$; (ii) 2, $\frac{38}{13}$.

2. (i) $\frac{(a^2 + b^2) + \sqrt{(a^2 + b^2)(3a^2 - b^2)}}{a^2 - b^2}$.

(ii) $\frac{-1 \pm \sqrt{-3}}{2}$, $\frac{a \pm \sqrt{a^2 - 4}}{2}$. 3. (i) $\pm 4a$. (ii) $-\frac{4}{5}$, $\frac{3}{5}$.

4. $\frac{5}{7}$. 5. (i) 1, 1, $\frac{3 \pm \sqrt{5}}{2}$, (ii) $-3 \pm \sqrt{5}$.

6. $x = 1$ or 2. 8. (i) 1, 2, 3, $\frac{-11 \pm i\sqrt{23}}{12}$. (ii) 1, $\frac{a-b}{b-c}$.

9. 15 ft. 10. 88'', 80''. 11. 16. 12. $\frac{1}{2}$, 5.

13. (i) $\frac{1}{3}$. (ii) $\frac{3 \pm \sqrt{5}}{2}$. (iii) 19. (iv) $\frac{(b-c)^2}{8a(b+c)}$.

(v) $a^2 + b$, $a - b$. (vi) $-(1 + 5a)$, $\frac{1 + 3a}{1 - 5a}$.

(vii) 1, $a - b - 1$. (viii) $\frac{b-c}{a}$, $\frac{c-a}{b}$.

(ix) $-2a$, $-a - b$. 14. $3(q + q') - pp'$.

15. (i) $5x^2 - 10x + 1 = 0$. (ii) $x^2 - \frac{2b^2 - 4ac}{a^2}x + \frac{b^4}{c^4} - \frac{4b^2}{ac} = 0$.
16. (a) $-p \pm \sqrt{p^2 + 1}$. (b) $x = -3, y = -2$; $x = 1, y = 2$.
17. $b^2 = ac$. 21. $p^2x^2 - (q^2 - 2rp)x + q^2 = 0$.
22. (i) 2, $\frac{1}{2}$. (ii) 4. 24. 27, -125.
27. $p^2 - 3q$. 28. -1. 30. $bb' = 2(ac' + a'c)$.
31. $(rc - pa)^2 = (qa - rb)(pb - qc)$. 32. $x^2 - x(a^2 - 2b) + b^2 = 0$.
33. $x = \frac{1}{2}[b \pm \sqrt{2a^2 - b^2}]$, $y = \frac{1}{2}[b \mp \sqrt{2a^2 - b^2}]$. 35. $\frac{1}{4}^6, -\frac{5}{2}$.
36. $x^2 - x(4a^2 - b)(4a^2 - 3b + 1) + (4a^2 - b)^2(4a^2 - 3b) = 0$.
37. (a) $x^2 - x - 1 = 0$. (b) $x^2 - 2x + 1 = 0$. 38. $k = 0$ and 3.
39. $y^2 - by + c = 0$. 40. 28. 41. $a = \pm 2$.

EXERCISE XX, Page 55

1. (i) 88. (ii) 43. (iii) $118 - 3n$. 2. -59. 3. 62.
4. 129. 5. $5n - 2$. 6. 29th, 4th. 7. 6th.
8. -2, 1, 4, 7, ... 9. -4, 1, 46, $5n - 9$. 10. 21.

EXERCISE XXI, Page 58

1. $\frac{n}{2}(3n + 7)$. 2. 10 or 4. 3. $\frac{n^2 - n - 2}{2}$.
4. $\frac{n-1}{2}$. 5. $\frac{1}{2}n(n-1)$. 6. 20. 7. (i) 10. 8. 10.
9. 2, 8, 14, 20, ..., 2, 5, 8, 11... 10. 24750. The answer -14 can be explained by the fact that 14 terms counted backwards from 10th term gives the same sum.
11. 11570. 12. 1, 5, 9, 13... 13. (i) $\frac{1}{2}n(3n-7)$.
- (ii) $\frac{1}{2}n(3n-23)$. 14. $a+b, 2a$. 15. $\frac{1}{2}(x^2 - 3y^2 - 2xy + y + 3x)$.
16. 9. 17. 15th Jan, 183 $\frac{3}{4}$ miles. 18. 200,000.

EXERCISE XXII, Page 62

1. $-2\frac{1}{2}, -1\frac{5}{8}, -1; -\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, 2\frac{1}{8}, 3\frac{1}{8}$. 2. $19\frac{1}{2}, 19, 18\frac{1}{2}, 18, 17\frac{1}{2}$.
3. $p^2 - p + 1, p^2 - 2p + 2, \dots, p$.
4. $4x - 3y, 3x - 2y, 2x - y, x, y, 2y - x, 3y - 2x, 4y - 3x$.
6. $n : m$.

EXERCISE XXIII, Page 65

1. 10. 2. 5, 8, 11. 3. 3, 5, 7. 4. 7, 9, 11.
5. 3, 5, 7, 9. 6. 1, 5, 9, 13. 7. 2, 6, 10, 14.
11. 8, 7, 11, 15..., it is an A. P. excluding the first term.
12. 3, 5, 7, 9, 11. 13. $p + q - m$.

EXERCISE XXIV, Page 69

1. $\frac{1}{2} \frac{8}{4} \frac{0}{3}$. 2. $\frac{8}{27}$; $(\frac{2}{3})^{n-1}$. 3. -1536. 4. 729.
 5. 10th. 6. 8th. 7. 5th. 8. (i) -114, 128, 246, 368.
 (ii) 2: 18, 54, 162, or -2, -18, -54, -162.
 9. 4, 8, 16, 32, 64, ...

EXERCISE XXV, Page 71

1. $73\frac{2}{2} \frac{2}{8} \frac{1}{6}$. 2. $-\frac{3}{2}(2-\sqrt{2})$. 3. $1-\frac{1}{10^n}$.
 4. 6. 5. -255. 6. $\frac{an(n+1)}{2} + \frac{x(1-x^n)}{1-x}$. 7. 353.8 ft.
 8. 6.

EXERCISE XXVI, Page 74

1. $\frac{1}{8}$. 2. $\frac{1}{3}$. 3. $6(\sqrt{2}+1)$. 4. $\frac{a^2}{a-1}$.
 5. $\frac{1}{1+2x}$. 6. $1\frac{5}{8}$. 7. $\frac{1}{3}, 3-\frac{1}{3^{n-1}}$.
 8. $\frac{2}{3}$. 10. x . 11. (i) $\frac{5}{9}$. (ii) $10\frac{1}{90}$. (iii) $\frac{2}{5} \frac{3}{9} \frac{1}{8}$.

EXERCISE XXVII, Page 77

1. 6, 12, 24. 2. $1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}$.
 3. -14, 7, $-\frac{7}{2}, \frac{7}{4}, -\frac{7}{8}, \frac{7}{16}$. 4. $-\frac{1}{2}$. 5. 2, 32. 6. $1, \frac{1}{84}$,
 or -1, $-\frac{1}{84}$. 7. 8 and 12 or $\frac{1}{2}$ and -3.

EXERCISE XXVIII Page 80

1. 2, 4, 8. 2. 3, 6, 12. 3. 4, 6, 9.
 4. 248. 5. 2, 5, 8 or 26, 5, -16.
 6. $\frac{a^{2n}-b^{2n}}{a^{2n-2}}$. 7. 3, -7. 8. ± 3 .
 9. (i) $\frac{(a+b)^3}{2a}$. (ii) $r = \pm \frac{2}{7}, 35, 10, 2\frac{6}{7}, \frac{40}{49}; 63, -18, 5\frac{1}{4}, -1\frac{28}{19}$.
 9. (b) 8 and 12 or $\frac{1}{2}$ and -3.
 10. $\frac{1}{81}[10^{n+1}-10-9n]$. 11. $\frac{1}{27}(9n-1+10^{-n})$.
 12. $n(n+1)+\frac{3}{2}(3^n-1)$. 17. $b-\frac{(b-a)^n}{b^{n-1}}$. 18. $1+\frac{1}{2}+\frac{1}{4}+\dots$
 19. $\frac{144}{5^2} + \frac{144 \times 7}{5^4} + \frac{144 \times 7^2}{5^6} + \dots$ 20. $d=2, r=3$
 or $d=10, r=-1$.
 21. $d=4, r=2; d=-\frac{4}{3}, r=-\frac{2}{3}$.

$$27. x\left(\frac{x}{y}\right)^{\frac{p-q}{p-q}} \quad 28. \frac{1}{x-y} \left\{ \frac{x^2-x^n+2}{1-x} - \frac{y^2-y^n+2}{1-y} \right\}.$$

$$29. 112 \text{ ft.} \quad 30. 19 \text{ ft.} \quad 31. 732. \quad 33. \frac{1}{(1-r)(1-br)}.$$

EXERCISE XXIX, Page 87

$$1. \frac{3}{17}. \quad 2. 7, \frac{7}{88}. \quad 3. \frac{4}{5}, \frac{6}{7}, 2, 8, -4. \quad 4. \frac{3}{3n-2}.$$

$$5. \frac{60}{16-n}. \quad 6. 15. \quad 7. \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{17}. \quad 8. 1.$$

$$9. 2, 6. \quad 10. 3, 12. \quad 12. \frac{1}{8}, \frac{1}{12}, \text{ or } 1, -\frac{1}{4}.$$

EXERCISE XXX, Page 90

$$1. \frac{1}{8}, \frac{1}{4}, \frac{1}{6}. \quad 2. \frac{1}{4}, \frac{1}{8}, \frac{1}{6}, \text{ or } \frac{1}{4}, \frac{1}{12}, -\frac{1}{6}. \quad 6. 104, 234.$$

$$9. \frac{ac(m-1)}{a(n-1)+c(m-n)}.$$

EXERCISE XXXI, Page 95

$$1. \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}. \quad 2. \frac{1+x}{(1-x)^2} - \frac{2x^n}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}.$$

$$3. n \cdot 2^n. \quad 4. 1\frac{1}{2}. \quad 5. \frac{1}{16}. \quad 6. \frac{2}{38}.$$

$$7. \frac{7}{4} - \frac{6n+7}{4 \cdot 3^n}, \quad \frac{7}{4}. \quad 8. (i) \frac{1+2x^2}{(1-x^2)^2}; \quad (ii) \frac{1}{(1-x)^2}.$$

$$9. \frac{1+x+x^2}{(1+x)^2}. \quad 10. \frac{1+7x}{(1-x)^2} + \frac{8x^2}{(1-x)^3}.$$

$$11. \frac{a^2}{(a-x)(a^2-x^2)}. \quad 13. 8$$

EXERCISE XXXII-A, Page 97

$$1. \frac{n(4n^2-1)}{3}. \quad 2. \frac{2n(n+1)(2n+1)}{3}. \quad 3. n^2(2n^2-1)$$

$$4. 2n^2(n+1)^2. \quad 5. 19270.$$

$$6. \frac{1}{8}\{n(n+1)(2n+1)-r(r+1)(2r+1)\}$$

EXERCISE XXXII-B, Page 101

$$1. (i) \frac{1}{6}n(n+1)(2n+7). \quad (ii) \frac{1}{2}n(n+1)(2n+3).$$

$$(iii) n(n+1)^2(n+2). \quad 2. \frac{p(4p^2+18p+23)}{3}.$$

$$3. n(n+1), \frac{1}{2}n(n+1)(n+2). \quad 4. \frac{1}{2}n(n+1)(n+2)(3n+5).$$

5. $\frac{1}{6}n(n+1)(n+2)$.
6. (i) $2n-2+\frac{1}{2^n-1}$. (ii) $\frac{1}{4}\left[6n-3+\frac{1}{3^n-1}\right]$.
7. $2^{n+1}-2-n$. 8. $\frac{1}{8}n(n^2+6n-1)$. 9. $\frac{1}{6}n(n+1)(n+2)$.
10. $\frac{1}{2}n^2(n+1)$. 11. $\frac{3}{2}n^2-\frac{1}{2}n+2$, $\frac{1}{2}n(n^2+n+4)$.
12. $\frac{3^n-1}{2}+2n$. 13. $\frac{1}{81}(10^{n+1}-10-9n)$.
14. $\frac{70}{81}(10^n-1)-\frac{7n}{9}$. 15. $2^{n+1}-2-n$.
16. (i) $\frac{n(n+1)}{2}+2^{n+1}-2$. (ii) $\frac{1}{6}n(2n^2+9n+1)$.
17. $\frac{3^{n+1}-2n-3}{4}$. 18. $\frac{n(n+1)(n+2)(n+3)}{4}$.
19. (i) $2n^3+3n^2+n$; $S_n=\frac{1}{2}n(n+1)^2(n+2)$.
(ii) 2^n-8 ; $S_n=2^{n+1}-2-8n$. 20. $\frac{n(n+1)^2(n+2)}{12}$.
21. $2n(n+2)^2$, $\frac{n(n+1)(3n^2+19n+32)}{6}$.
22. $\frac{n}{4}(n+1)(n+2)(n+3)$. 23. $\frac{n}{2n+1}$.
24. $\frac{1}{15}-\frac{1}{3(3n+5)}$. 25. $\frac{2n}{n+1}$.
26. $5-\frac{1}{3^{n-2}}$; $\frac{1}{2}\left(10^n-9+\frac{1}{3^{n-2}}\right)$.

EXERCISE XXXIII, Page 104

1. 5. 120 2. 1240. 3. 220. 4. 5525.
5. 348.

REVISION QUESTIONS II, Page 105

1. $\frac{n}{2}[(n+5)a-(3n+1)b]$. 2. $\frac{n^2-n+1}{n}$.
3. (i) $\frac{n(n+139)}{200}$. (ii) $\frac{n-1}{2}$. (iii) $\frac{n(n-1)}{2}$.
4. 4 or 10. 6. 1, 3, 5, 7. 7. 1, 2, 4, 8.
10. 9, 16. 11. 16.
12. (i) $\frac{3\sqrt{3}}{8}$. (ii) $\frac{1}{1-x}$. (iii) $\frac{13}{24}$. (iv) 8.

14. If a and b are the quantities, then $\frac{a}{b} = \frac{n^2 - 2 \pm n\sqrt{n^2 - 4}}{2}$.
15. 16. 18. (i) $\frac{60}{16-n}$ (ii) $\frac{20}{5n+3}$ 22. $\frac{n(n+1)(3n-2)}{3}$.
24. (i) $\frac{2n^3 + 3n^2 + 7n}{6}$ (ii) $2^{n+1} - 2 - n$ (iii) $\frac{1-x^{100}}{(1-x)^2} - \frac{100x^{100}}{1-x}$.
25. (i) $\frac{1}{1+x} + \frac{x^2}{(1+x)^2}$ (ii) $\frac{1+x-x^2}{(1-x)}$ (iii) $-n(2n+1)$.
26. $\frac{7}{8}$ 27. $\frac{1}{2}[(a - \frac{1}{2}d) - (a + nd - \frac{1}{2}d)(-1)^n]$.
28. $\frac{-n(n+2)}{4}$ (n even) or $\frac{1}{4}(n+1)^2$ (n odd).
29. $\left[\frac{n(n+1)}{2} \right]^2$ 30. $\frac{1}{6}n(n+1)(n+2)$.
34. $\frac{r(r^p-1)}{(r-1)^3} - \frac{ap}{r-1}$ 35. r 42. $n=0, -\frac{1}{2}, -1$.
44. 21. 45. 1956.

EXERCISE XXXIV-A, Page 117

1. (i) 7. (ii) 6. (iii) 18. (iv) 11, 5, 120, 60, 6, 21.
6. 24, 6, 6, 6. 7. (i) 60. (ii) 120. 8. 3024. 9. 120.
- (i) 24. (ii) 48. (iii) 24 (iv) 48. 10. 2640. 11. 18, 8.
12. 120. 18. 40320, 5040. 14. 325. 15. 1440. 16. 3600.

EXERCISE XXXIV-B, Page 120

1. (i) 24. (ii) 256. 2. 625. 3. $|n|; n^n; n^n - n$.
4. 4^n . 5. 30. 6. 155

EXERCISE XXXIV-C, Page 122

1. (i) 560. (ii) 5040. (iii) 151200. (iv) 4989600.
2. 180, 12, 18. 8. 420, 360. 4. 48. 5. 1680.

EXERCISE XXXV, Page 127

1. 55, 55, $\frac{1}{6}n(n-1)(n-2)$, $\frac{2}{(\lfloor n \rfloor)^2}$.
2. (i) 3. (ii) 28. 5. $n=34, r=14$. 6. 120.
7. 455, $\frac{1}{6}n(n-1)(n-2)$. 8. 20, 54, $\frac{1}{6}n(n-3)$. 10. 870.
11. $n-{}^nC_{r-1}$. 12. 210, 84. 13. 252, 53. 14. 56, 21. 17. 36.
18. 990. 19. (i) 36. (ii) 225.

EXERCISE XXXVI-A Page 130

1. 19900, 78, 1. 2. 161700. 3. 1140. 4. 465, 528.
 5. 300, 351. 6. $n-1$. 7. 20, 8. 8. 80, 7.
 9. 7.

EXERCISE XXXVI-B, Page 132

1. $\frac{\overline{100}}{\overline{40} \overline{60}} ; \frac{2\overline{100}}{\overline{40} \overline{60}} . 2. \frac{\overline{15}}{(\overline{5})^3} . 3. \frac{\overline{9}}{(\overline{3})^3} . 4. 27720.$
 6. 369600.
 7. $\frac{\overline{52}}{(\overline{13})^4} . 8. \frac{\overline{12}}{\overline{2} \times (\overline{6})^2} ; \frac{\overline{12}}{\overline{3} \times (\overline{4})^3} .$
 9. $\frac{\overline{30}}{\overline{3} (\overline{10})^3} ; \frac{\overline{30}}{(\overline{10})^3} .$

EXERCISE XXXVI-C, Page 138

1. 18. 2. 136. 3. (a) 1260, 4989600 (b) 270, 2454.
 4. 59, 779. 5. 136, 2454

MISCELLANEOUS EXAMPLES Page 138

8. 3780. 4. 6300. 5. (i) 280. (ii) 736.
 7. 145. 8. (i) $\frac{1}{2}p(p-1) - \frac{1}{2}q(q-1) + 1$.
 (ii) $\frac{1}{8}[p(p-1)(p-2) - q(q-1)(q-2)]$.
 12. 10080. 18. 1728.
 14. (i) 10080. (ii) 30240. (iii) 4320. (iv) 1440.
 (v) 720. (vi) 1440.
 15. 24, 48, 240, 480, 48, 16. 16. (i) 40. (ii) 8.
 17. 36. 18. 14400. 19. 12. 20. 1716.
 24. (i) 126. (ii) 56. 29. 5040. 30. 3. 31. (a) 9.
 (b) $\overline{9} \div \overline{2} = 181440$. 32. 181440. 33. $\overline{9} \times \overline{3}$.
 34. $\overline{8} \times \overline{7}$ 36. $\frac{\overline{13}}{40}$ 37. (i) 462. (ii) 42.

EXERCISE XXXVII, Page 148

1. 3600. 2. 36. 3. 2002. 4. 5936. 5. 55.
 (i) 336. (ii) 672 7. 246. 8. 41. 9. 1000.
 12. $4^5 - 1 = 1023$. 13. 105. 14. 144.
 15. (i) 2880. (ii) 1440. 16. (i) 576. (ii) 288.
 17. 126. 21. $n(n-1)(n-2) \dots (n-m+1), m^n$.

REVISION QUESTIONS III, Page 151

$$1. (i) {}^{n-2}C_{r-2} \times \underline{r} \quad \text{or} \quad {}^{n-2}P_{r-2} \times {}^rP_2 = \frac{|n-2}{n-r} \times r(r-1).$$

$$(ii) (n-2)(n-3) \dots (n-r-1).$$

2. 1440. 5. 60. 12. 10 13. 35.
 14. 35. 15. 113400. 16. 32. 17. 48.
 20. 3999960. 21. 133320. 22. 360. 28. 72.
 24. 810

EXERCISE XXXVIII, Page 158

$$1. (i) 1+4x+6x^2+4x^3+x^4 \quad (ii) 1-5x+10x^2-10x^3+5x^4-x^5.$$

$$2. (i) 64x^5-64x^5y+\frac{80}{3}x^4y^2-\frac{160}{27}x^3y^3+\frac{20}{27}x^2y^4-\frac{4}{81}xy^5+\frac{y^6}{729}$$

$$(iii) \frac{x^6}{y^6}+6\frac{x^4}{y^4}+15\frac{x^2}{y^2}+20+15\frac{y^2}{x^2}+6\frac{y^4}{x^4}+\frac{y^6}{x^6}.$$

$$3. (i) \frac{155584}{27}x^7a^{10}. \quad (ii) 760ab^6. \quad 4. (a) 1890a^4 \quad (b) (-1)^n {}^{2n}C_n.$$

$$5. (i) \frac{|110}{(|55)^2}x^{55}. \quad (ii) 1120a^4x^4.$$

$$6. (i) \frac{189}{8}x^{17}, -\frac{21}{16}x^{19}. \quad (ii) -112266x, \frac{336798}{x}.$$

$$(iii) \frac{|2n+1}{|n|n+1} a^{n+1}x^n, \frac{|2n+1}{|n|n+1} a^nx^{n+1}.$$

$$7. p=-\frac{1}{2}, n=6.$$

$$8. 1st, 3rd., 5th., 7th.$$

$$9. -\frac{1}{8}x.$$

$$10. n=11, x=2.$$

EXERCISE XXXIX Page 161

$$1. (i) \text{No.} \quad (ii) 210a^{16}b^6. \quad 2. \frac{|36}{|5|31} 2^{31}.$$

$$3. -99 \times 2^{10}. \quad 5. (i) -45. \quad (ii) -252. \quad 6. -462x, \frac{462}{x}.$$

$$7. 1120. \quad 8. 36. \quad 9. (i) 1890b^4. \quad (ii) 7920. \quad (iii) 5376.$$

$$(iv) 252. \quad (v) \frac{7}{15}. \quad 10. 210; 0. \quad 12. 5. \quad 14. 770. \quad 15. -640.$$

$$16. a=\frac{2}{3}, n=7. \quad 17. a=\frac{1}{2}, n=16. \quad 18. (2+3)^6. \quad 19. 997002999.$$

EXERCISE XL, Page 165

$$1. (n-8)\text{th}; (n-r+2)\text{th}; (n-r+1)\text{th}.$$

$$2. 2^{14}-1. \quad 4. 2^{30}. \quad 5. 2^{17}. \quad 6. (n+2)2^{n-1}, \frac{2^{n+1}-1}{n+1}.$$

$$8. \frac{\frac{2n}{n-1}}{n+1}$$

$$9. \frac{\frac{2n}{n-2}}{n+2}$$

$$10. \frac{\frac{n}{r-1}}{n-r+1} y^{n-r+1} (2x)^{r-1}; \frac{\frac{n}{r-1}}{n-r+1} y^{r-1} (2x)^{n-r+1}.$$

$$11. \frac{\frac{3n}{2n}}{n} \frac{1}{x^n} \quad 13. {}^nC_r y^r x^{n-r}; \frac{\frac{2n}{n}}{n}$$

EXERCISE XLI, Page 167

1. (i) 6th. (ii) 9th. (iii) 6th and 7th. 2. (i) 4th. (ii) 5th.
3. (i) 3rd and 4th. (ii) 5th. (iii) 4th and 5th. 4. 4th.

EXERCISE XLII, Page 172

$$1. (i) x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}}a - \frac{1}{9}x^{-\frac{4}{3}}a^2 - \frac{4}{81}x^{-\frac{7}{3}}a^3.$$

$$(ii) a^{\frac{2}{3}} - \frac{2}{3}a^{-\frac{1}{3}}x^4 - \frac{1}{3^2}a^{-\frac{5}{3}}x^8 - \frac{1}{1^2 \cdot 3}a^{-\frac{8}{3}}x^{12}.$$

$$(iii) \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^5} + \frac{1}{x^7}. \quad (iv) 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \frac{35}{8}x^4.$$

$$2. (i) 16 + \frac{8x}{3} + \frac{1}{18}x^2 - \frac{1}{648}x^3 \dots\dots$$

$$(ii) x^{\frac{4}{9}} + \frac{32}{3}x^{\frac{1}{3}} + \frac{128}{9}x^{-\frac{2}{3}} - \frac{2048}{81}x^{-\frac{5}{3}} \dots\dots$$

$$3. \frac{110}{243}a^{-11}.$$

$$4. (-1)^r \frac{1 \cdot 3 \cdot 5 \dots (2r-1)}{r} x^r, x < \frac{1}{2} \text{ numerically.}$$

$$5. (i) -\frac{208}{9}x^7.$$

$$(ii) -\frac{399}{5}x^{10}.$$

$$(iii) -\frac{14}{729}x^6.$$

$$(iv) \frac{-13}{6912}x^5.$$

EXERCISE XLIII, Page 175

$$1. 4 + \frac{29}{48}x.$$

$$2. 2 + \frac{5}{2}x.$$

$$3. 1 - \frac{17}{12}x.$$

$$4. (i) 1 - \frac{35}{24}x.$$

$$(ii) 1 - \frac{1}{6}x.$$

$$5. 2 - \frac{17}{4}x.$$

$$6. 5.775, 7.835, 5.273.$$

$$7. .03 \text{ cubic feet.}$$

$$10. .003 \text{ cm.}$$

$$12. 1.127.$$

$$13. (i) 5.00096.$$

$$(ii) 1.05882.$$

$$(iii) 1.00067.$$

$$14. 1.27442.$$

$$15. 127.1067.$$

$$16. 10.0067, 5.0133.$$

$$18. 1.25992.$$

$$19. 1.25992 \text{ (approx.)}$$

$$20. 9.993.$$

EXERCISE XLIV, Page 179

1. (i) Third, positive. (ii) All terms positive.
 (iii) Thirteenth ; positive. 2. (i) $8n-4$. (ii) $2n^2+2n+1$.
 3. (i) $3n+1$. (ii) $(-1)^n(n^2+2n+2)$. (iii) $(-1)^n(n^3+3n)$.
 6. $\frac{n(n+1)\dots(n+r-1)}{r} x^r ; x < 1$. 7. (i) 1. (ii) $(-1)^n$.
 8. (i) Third term. (ii) Sixteenth and seventeenth terms.
 9. (i) 2. (ii) $\sqrt{2}$. (iii) $\frac{5}{7}\sqrt{2}$.

REVISION QUESTIONS IV, Page 184

3. $1-2x-2x^2-4x^3\dots$ 4. 252.
 5. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots$ 6. $\left(\frac{5}{4}\right)^7 \frac{3.7.11.15.19.23.27}{7}$
 7. $\frac{(r+1)(r+2)(r+3)x^r}{1.2.3}$, 1. 8. n . 10. 2.
 11. The co-efft. of x^0 is 1 not 0. 12. ${}^{20}C_8$, 2^8 , 3^{12} ; $(-1)^{n-1}$.
 13. $1+\frac{1}{2}x+\frac{3}{8}x^2-\frac{3}{16}x^3+\frac{3}{128}x^4+\frac{15}{256}x^5+\dots$
 15. 1.1705. 16. (i) 10.0332. (ii) 9.99333.
 (iii) 3.9579, .2527. (iv) 10.00999 approx. (v) 3.002.
 (vi) 4.9959952. 17. (i) 996005996001. (ii) 2.002.
 19. $\frac{2r+1}{(r)^2} \cdot \frac{x^r}{2^r} ; 1.0099$.
 20. (a) 2^r (take $n=-r$ and $x=-1$). (b) 1.00039928.
 21. 4.89893950.

EXERCISE XLV, Page 188

1. $\frac{3}{x-2}-\frac{4}{x-3}$ 2. $\frac{4}{3x+1}+\frac{1}{x+1}$
 3. $\frac{1}{3} \cdot \frac{1}{(x-1)}+\frac{5}{3} \cdot \frac{1}{x+2}+1$
 4. $\frac{3}{x+3}-\frac{1}{x+2}-x$ 5. $\frac{1}{x}+\frac{2}{x+1}-\frac{4}{2x+1}$
 6. $3+\frac{13}{4} \cdot \frac{1}{x-3}-\frac{9}{4} \cdot \frac{1}{x+1}$ 7. $1-\left(\frac{3}{x+1}-\frac{3}{2x+1}\right)$
 8. $x+3+\frac{5}{2x-3}-\frac{2}{x+2}$ 9. $-\frac{2}{1-2x}-\frac{1}{2} \cdot \frac{1}{1+x}+\frac{1}{2} \cdot \frac{1}{1-x}$

10. $\frac{1}{2x-1} - \frac{1}{2x-3} + \frac{3}{5x-5}$ 11. $\frac{1}{6(x+1)} - \frac{1}{2(x-1)} + \frac{4}{3(x-2)}$
12. $\frac{a^2}{(a-b)(a-c)(x-a)} + \frac{b^2}{(b-a)(b-c)(x-b)} + \frac{c^2}{(c-a)(c-b)(x-c)}$
13. $\frac{1}{x-a+b} + \frac{1}{x-a-b} - \frac{2}{x-a}$
14. $\frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x+2}$ 15. $\frac{1}{2(x-1)} - \frac{4}{(x-2)} + \frac{9}{2(x-3)}$
16. $-\frac{1}{2} \frac{1}{(x-1)} - \frac{1}{2} \frac{1}{(x-2)} + \frac{1}{2} \frac{1}{x-3} + \frac{1}{2} \frac{1}{x-4}$
17. $\frac{2}{x^2+3} - \frac{1}{x^2+2}$

EXERCISE XLVI, Page 191

1. $\frac{1}{(1-x)^2} - \frac{1}{1-x}$ 2. $\frac{9}{(1-x)^3} - \frac{5}{(1-x)^2}$
3. $\frac{1}{4} \left\{ \frac{1}{(x+1)^2} + \frac{1}{(x+1)} + \frac{1}{(x-1)^2} - \frac{1}{(x-1)} \right\}$
4. $\frac{1}{x-2} + \frac{3}{2(x+1)} - \frac{5}{2(x+1)^2}$
5. $\frac{3}{2(x-1)^2} + \frac{3}{4(x-1)} + \frac{1}{4(1+x)}$
6. $\frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3} + \frac{1}{(x-1)^4}$
7. $\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2} + 1$ 8. $\frac{2}{x-1} + \frac{3}{(x-1)^2} - \frac{6}{3x-4}$
9. $\frac{1}{2} \frac{1}{(x+1)^2} - \frac{1}{4} \frac{1}{x+1} + \frac{1}{4} \frac{1}{(x-1)}$
10. $\frac{1}{3(x-1)} + \frac{5}{3(x+2)} - \frac{3}{(x+2)^2}$
11. $\frac{1}{(x-1)^2} + \frac{5}{3} \cdot \frac{1}{(x-1)^2} + \frac{13}{9} \cdot \frac{1}{x-1} + \frac{5}{9} \cdot \frac{1}{x+2}$
12. $-\frac{1}{x^2} - \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x-1}$
13. $\frac{1}{3} \frac{1}{x+3} - \frac{1}{3x} + \frac{1}{x^2} - \frac{2}{x^3}$

$$14. \frac{1}{x^4} - \frac{1}{x^3} + \frac{1}{x^2} - \frac{1}{x} + \frac{1}{1+x}.$$

$$15. \frac{8}{9} \frac{1}{(x-2)^3} + \frac{20}{27} \frac{1}{(x-2)^2} + \frac{2}{27} \frac{1}{(x-2)} + \frac{1}{27} \frac{1}{(x+1)^2} - \frac{2}{27} \frac{1}{x+1}$$

$$16. \frac{10}{x-3} - \frac{10}{x-2} - \frac{9}{(x-2)^2} - \frac{5}{(x-2)^3}.$$

$$17. \frac{5}{6} \frac{1}{(2x-1)^2} + \frac{1}{18} \frac{1}{2x-1} + \frac{2}{9} \frac{1}{x+1}.$$

$$18. \frac{1}{2(x+1)^2} + \frac{3}{4(x+1)} - \frac{1}{x} + \frac{1}{4(x-1)}$$

$$19. \frac{1}{2} \frac{1}{(x^2+1)^2} - \frac{1}{4} \frac{1}{1+x^2} + \frac{1}{4} \frac{1}{x^2+3}.$$

$$20. \frac{a^2}{a^2-b^2} \frac{1}{(x^2+a^2)^2} + \frac{b^2}{(a^2-b^2)^2} \frac{1}{x^2+a^2} - \frac{b^2}{(a^2-b^2)^2} \frac{1}{x^2+b^2}.$$

$$21. \frac{1}{(x-1)^2} + \frac{1}{x-1} + \frac{1}{x+3}.$$

EXERCISE XLVII, Page 194

$$1. \frac{3}{1+x} + \frac{4-3x}{1+x^2} \quad 2. \frac{1}{3} \frac{1}{1-x} + \frac{1}{3} \frac{x+2}{x^2+x+1}$$

$$3. \frac{1}{8(x-2)} - \frac{x-6}{8(x^2+4)} \quad 4. \frac{3}{2(x-1)} + \frac{1-3x}{2(1+x^2)}$$

$$5. \frac{1}{6} \frac{1}{(x-1)} - \frac{1}{2} \frac{1}{x+1} + \frac{1}{3} \frac{x-1}{1+x+x^2}$$

$$6. \frac{x}{x^2+1} - \frac{1}{x+1} \quad 7. \frac{x}{2+x^2} - \frac{x}{3+x^2}$$

$$8. \frac{7x+9}{2(x^2+2x+5)} - \frac{1}{2(x+1)} \quad 9. \frac{1}{2(x-1)^2} - \frac{1}{2(x-1)} + \frac{x}{2(x^2+1)}$$

$$10. \frac{1}{(x-2)^2} - \frac{1}{x-2} + \frac{x-1}{x^2-x+1}$$

$$11. \frac{1}{12} \left[\frac{1}{x+2} + \frac{2}{x-1} + \frac{3-3x}{x^2-x+2} \right]$$

$$12. \frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x^2+1} \quad 13. \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}$$

$$14. \frac{x+1}{20(x^2+x+1)} - \frac{1}{20(x-2)} + \frac{1}{10(x-2)^2}$$

15. $1 + \frac{1}{x} + \frac{2x+4}{x^2+x+1}$ 16. $\frac{1}{2} \cdot \frac{1}{x^2+x+1} + \frac{1}{2} \cdot \frac{1}{x^2-x+1}$
17. $\frac{1}{2\sqrt{2}} \left\{ \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1} \right\}$
18. $\frac{1}{(b^2-a^2)(x^2+a^2)} + \frac{1}{(a^2-b^2)(x^2+b^2)}$
19. $\frac{x}{x^2+x+2} - \frac{1}{x^2+x+1}$
20. $\frac{1}{a^3} \left\{ \frac{1}{x-a} - \frac{2}{x} - \frac{a}{x^2} + \frac{x}{x^2+a^2} \right\}$
21. $\frac{1}{2} \cdot \frac{x+1}{(1+x^2)^2} + \frac{1}{4} \cdot \frac{x-1}{1+x^2} - \frac{1}{4} \cdot \frac{1}{1+x}$
22. $\frac{x-1}{x^2+1} + \frac{1}{(x^2+1)^2} - \frac{x-1}{x^2+2}$
23. $\frac{4}{9(x+1)^2} + \frac{5}{9(x^2-x+1)} + \frac{x}{3(x^2-x+1)^2}$
24. $\frac{i}{4} \frac{1}{(x+i)^2} - \frac{i}{4} \frac{1}{(x-i)^2}$
25. $x + \frac{1}{4(x-1)} + \frac{1}{4(x+1)} - \frac{x}{2(x^2+1)}$;
 $\frac{1}{9(x-1)} + \frac{2}{9(x+2)} - \frac{x}{3(x^2+x+1)}$

EXERCISE XLVIII, Page 196

1. $1 - \frac{1}{n+1}$; 1. 2. $\frac{n}{3n+1}, \frac{1}{3}$
3. $\frac{2^n}{1+2^n x} - \frac{1}{1+x}; \frac{1}{x(1+x)}$ 4. $\frac{n^2}{(n+1)(n+2)}$
5. $\frac{11}{18} - \frac{1}{3} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \right)$
6. $-\frac{3}{2} - \frac{1}{2}(-1)^n$ 7. $-\frac{1}{3} - \frac{2}{3}(-1)^n 2^n$
8. $-\frac{1}{4} \left[3 + \frac{5}{3^8} \right]$ 9. $-1 - (-1)^n$
10. $\frac{3}{1+x^2} + \frac{4}{1-x}; 7, 1.$

REVISION QUESTIONS V, Page 196

1. $\frac{1}{3(x+2)} + \frac{2}{3(x-1)}$ 2. $\frac{1}{x^2} + \frac{1}{1+x} - \frac{1}{x}$ 8. $\frac{2}{x^2+1} - \frac{1}{x^2+2}$
4. $x - \frac{1}{6} \cdot \frac{1}{x-1} - \frac{1}{6} \cdot \frac{1}{x+1} + \frac{8}{3} \cdot \frac{1}{x-2} + \frac{8}{3} \cdot \frac{1}{x+2}$
5. $-\frac{3}{25} \cdot \frac{1}{x+2} + \frac{3}{25} \cdot \frac{1}{x-3} + \frac{7}{5} \cdot \frac{1}{(x-3)^2}$
6. $\frac{1}{2} \cdot \frac{x+1}{x^2+x+1} - \frac{1}{2} \cdot \frac{x-1}{x^2-x+1}$
7. $\frac{1}{(a^2-b^2)} - \frac{a^2x}{x^2+a^2} - \frac{1}{(a^2-b^2)} - \frac{b^2x}{x^2+b^2}$
8. $\frac{1}{4} \cdot \frac{1}{1+x} + \frac{1}{4} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{1+x^2}$
9. $\frac{1}{1+x^2} - \frac{1}{x^2} + \frac{1}{x^4}$ 10. $\frac{1}{2} \cdot \frac{2x-1}{x^2-x+1} - \frac{1}{2} \cdot \frac{2x+1}{x^2+x+1}$
11. $\frac{1}{x} - \frac{x}{1+x^2} - \frac{x}{(1+x^2)^2} - \frac{x}{(1+x^2)^3}$
12. $-\frac{9}{25} \cdot \frac{1}{x+2} - \frac{1}{5} \cdot \frac{x-2}{(x^2+1)^2} + \frac{9x+7}{25(x^2+1)}$
18. (i) $\frac{1}{8} \cdot \frac{1}{x-2} - \frac{1}{8} \cdot \frac{x-6}{x^2+4}$
 (ii) $\frac{3}{16(x-2)} - \frac{3}{16(x+2)} + \frac{1}{4(x+2)^2}$
14. (i) $\frac{1}{27(x+2)} - \frac{1}{27(x-1)} + \frac{1}{9(x-1)^2} + \frac{2}{3(x-1)^3}$
 (ii) $\frac{1}{2(x-1)} - \frac{1}{10(x+1)} + \frac{2(1-x)}{5(x^2+4)}$
15. $\frac{1}{1+x} - \frac{1}{n+1+x}$ 16. $\frac{1}{1+x}$ 17. $4r^2+r+1$
18. (i) $-\frac{1}{4}$ (ii) $\frac{r+4}{2^{n+1}}$ 19. $1+(-1)^n$
20. $(-1)^n(2^{n+1}+2n+1)$ provided $-\frac{1}{2} < x < \frac{1}{2}$.

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1. (i) $4\frac{1}{2}$ (ii) $\frac{9}{18}, \frac{4}{18}$ 8. 997002999 4. $x=11, y=8$
9. $m=2, \frac{-10}{y}$ 10. (i) 1, -9, $-4 \pm 2\sqrt{-5}$
 (ii) $-1 \pm \sqrt{7}, -1 \pm \sqrt{23}$ (extraneous roots).

11. (i) $abx^2 - (a+b)x + 1 = 0$. (ii) $x^2 - 2x + 3 = 0$.
13. (a) $\frac{7}{18}$. (b) 1120. (c) $-462x, \frac{462}{x}$.
16. (i) $x = -\frac{3}{2}$. (ii) 1. 19. 0, $\frac{13}{5}$.
22. $7(2 - \sqrt{3}), 7, 7(2 + \sqrt{3})$. 28. (i) $1890a^4$. (ii) $n=11, x=2$
24. 210; 84 25. $(-1)^n 8^n c_n$. 26. (i) 64, 27.
- (ii) $x=15, y=2; x=-2, y=-15$.
27. $p^3x^3 - pq^2x + (q^2 - 2rp)2r = 0$. 28. $\frac{10500}{x^3}$.
29. (i) 27, $\frac{25}{147}$. 30. $\frac{4n}{3}$ th term.

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2. (b) (i) $\frac{5}{8}[10^n + 1 - 9n - 10]$. (ii) $\frac{2}{3}$.
3. (a) 19600. (b) 56. 4. (a) 6, $-5\frac{2}{3}, 1 \pm \frac{\sqrt{37}}{6}$.
- (b) $\frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}$.
5. (a) 15. (b) $2\sqrt{2-1}$.
- 1950
1. (a) $(ca^1 - c^1a)^2 = (bc^1 - b^1c)(ab^1 - a^1b)$.
- (b) (i) $\frac{-14}{5}$. (ii) $x=3, y=2; x=-2, y=-3$.
2. (b) $\frac{1}{(1-r)(1-br)}$. 8. (b) 1800.
4. (i) $\frac{1}{12} \left[\frac{1}{x+2} + \frac{2}{x-1} + \frac{3-3x}{x^2-x+2} \right]$.
- (ii) $\frac{-1}{x} + \frac{1}{4(x-1)} + \frac{3}{4(x+1)} + \frac{1}{2(x+1)^2}$.
5. (b) $n 2^{n-1}$

1951

1. $\frac{3p+1}{2} - \frac{3}{2} - p$. 2. (i) $x=4$.
- (ii) $x = \pm \sqrt{\frac{153}{8}}, y = \pm \sqrt{\frac{17}{8}}; x = \pm \sqrt{\frac{136}{7}}, y = \pm \sqrt{\frac{17}{14}}$
3. 360. 4. $-\frac{10}{3}$.

$$5. (i) \frac{1}{5(x+2)} - \frac{1}{10(x+3)} + \frac{x+1}{10(x^2+1)},$$

$$(ii) -\frac{1}{(x-1)^3} - \frac{2}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{x-2}.$$

1952

$$1. (i) x = \frac{4}{13}, \frac{9}{13}.$$

$$(ii) x=3, 6 \\ y=10, 5.$$

$$2 \quad 185.$$

$$3. \quad 4 \quad 1329793.$$

$$4. (i) \frac{1}{3(x-1)} + \frac{1}{3} - \frac{1-x}{x^2+x+1},$$

$$(ii) \frac{1}{(x-2)^2} + \frac{3}{5(x-2)} + \frac{2}{5(x+3)}.$$

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1949

$$1. (a) 4, 5, -\frac{1}{3} \text{ (Extraneous)} \quad (b) \frac{1}{(x-1)^2} + \frac{1}{x-1} + \frac{1}{x+3}.$$

$$(c) \pm(\sqrt{3} + i\sqrt{2}).$$

$$2 \quad (b) y^2 + y(2q - p^2) + q^2 = 0.$$

$$(c) (ca^1 - ac^1) = (ab^1 - a^1b)(bc^1 - cb^1).$$

$$3. (a) \frac{n(n+1)}{2},$$

$$(b) \frac{8}{81} [10n + 1 - 9n + 10].$$

$$(c) \frac{1-n^n}{(1+x)^2} - \frac{nx^n}{1-x}.$$

$$4. 15 \quad (i) 462.$$

$$(ii) {}^{11}C_7 + {}^{11}C_8 + {}^{11}C_9 + {}^{11}C_{10} + {}^{11}C_{11}.$$

$$5. (a) 31.594.$$

$$(b) {}^{15}C_7, {}^{30-9}, 105, {}^{15}C_4.$$

1950

$$1. (i) -9, 1, -4 \pm \sqrt{-20}.$$

$$(ii) \frac{9}{2}, \frac{-29}{14} \text{ (Both extraneous).}$$

$$(iii) x=9, y=7; x=7, y=9. \quad 2. (b) 841x^2 - 174x - 40 = 0.$$

$$(c) x=1, y = \frac{2ab}{a^2+b^2}.$$

$$3. (c) \frac{n}{3}(2n^2-1).$$

$$4. 18. \quad 5. (a) \frac{7}{18}, (b) 3.005, (c) \frac{3}{2(x-1)} + \frac{1-3x}{2(x^2+1)}.$$

1951

$$1. (i) \frac{4}{13}, \frac{9}{13}, \text{ and } x = \pm\sqrt{3}, y = \pm\sqrt{2}. \quad (ii) 4, -\frac{1}{8}, \quad (iii) y=0, x = \pm 1$$

$$2. (c) \pm(\sqrt{7} - 2\sqrt{-1}). \quad 3. (c) \frac{1}{(1-x)^2}.$$

4. (b) 12.

5. (a) $\frac{189a^{17}}{8}, -\frac{21a^{19}}{16}.$

5. (c) $\frac{1}{2(x-1)^3} + \frac{3}{4(x-1)^2} + \frac{1}{8(x-1)} - \frac{1}{8(x+1)}.$

1952

1. (b) 5, 3.

(b) $p^3 - 3pq.$

2. (i) 5, (ii) $-1 \pm 4\sqrt{-1}, -1 \pm \sqrt{80}.$ (iii) $x=6, y=4$

8. (b) 2. (c) $\frac{n(2n-1)(2n+1)}{3}.$ 4. (c) 230.

5. (c) $\frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}.$

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1946

1. (a) (i) $1, \frac{1}{2}, \frac{1}{5}.$ (ii) $x=2, y=1; x=-1, y=-2.$

(b) 36 or 40. 2. (b) $\frac{2n}{n+1}.$

8. (b) (i) 825. (ii) 1490.

4. (a) $x + \frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)}.$ (b) 10.0483.

1947

1 (i) $-\frac{2}{3}, \frac{1 \pm \sqrt{10}}{3}.$ (ii) -2 (extraneous). $\frac{1}{3}$ (extra).

2. $x^2 - 2px + q^2 - q = 0.$

4. (i) 5. (ii) $2^n + n; 2^{n+1} + \frac{1}{2}n(n+1) - 2.$

5. (a) 14. (b) $\frac{2n(2n-1)(2n-2)\dots\left(-\frac{2n}{7} + \frac{2r}{7} + 1\right)}{1 \frac{2}{7} (8n-r)}$

1948

1. (i) $\frac{9}{2}$ (extraneous, $-\frac{3}{2}$ (extraneous).

(ii) $x = \pm 1, y=0, x=\sqrt{3}, y=\pm\sqrt{2};$

$x = -\sqrt{3}, y = \pm\sqrt{2}.$ 2. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$

3. $t_n = 2n^2 + n + 3; S_n = \frac{n}{6} (4n^2 + 9n + 23).$

4. 4096. 5. (b) $\frac{1}{2(x^2-x+1)} - \frac{1}{2(x^2+x+1)}.$

